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Controlling phonons and photons at the wavelength scale: integrated photonics meets integrated phononics: supplementary material

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A. THEORETICAL DESCRIPTION

We first give a description of 3D-confined cavity optomechanics. Next, we connect to the description of 2D-confined waveguides.

A. Cavities: 3D-confined

We focus on cavity optomechanics [1] in this section although much of it applies to electromechanics as well. The dynamics of an optomechanical system involves taking into account the interplay between the coupled acoustic and optical degrees of freedom in a system. The frequencies of these two coupled degrees of freedom are typically different by many orders of magnitude so that the only physical significant coupling arises parametrically in the form described below. As a first step, a modal decomposition of time varying deformations in the elastic structure is considered, so a set of parameters $x_j(t)$, each encoding the deformation due to a particular vibrational mode is considered. A similar decomposition of Maxwell's equations leads to a set of electromagnetic modes of the structure with amplitudes $\alpha_j(t)$, which with the correct normalization would lead to $U_j = \hbar\omega_o|\alpha_j|^2$ being the energy and $|\alpha_j|^2$ the average photon number in mode j. Each optical (mechanical) mode of the structure has a frequency ω_o (ω_m) and their associated dynamics. At first we will only consider the interaction between two modes: a single optical and a single mechanical mode of the structure. Optomechanical interactions give rise to coupling between these modes in the following way: the deformation of the structure in a specific vibrational mode parametrized by x, causes a change in the optical frequency given by $\omega_o(x) = \omega_o(0) + Gx$, where the optomechanical coupling parameter $G = \partial_x\omega_o$ has units of $\mathrm{Hz}\cdot\mathrm{m}^{-1}$. The modal equation for the electromagnetic field, under laser excitation at frequency $\omega_{in} = \omega_o(0) - \Delta$ with input photon flux given by $|\alpha_{in}|^2$ is then expressed as

$$\frac{d\alpha(t)}{dt} = -\left(i(\Delta + Gx) + \frac{\kappa}{2}\right)\alpha - \sqrt{\kappa_{ex}}\alpha_{in}. \quad (1)$$

The cavity decay rate κ represents the full-width half-maximum (FWHM) of the optical mode excitation spectrum and contains all decay channels coupling to the photonic system. Typically, κ consists of an engineered *extrinsic* decay rate κ_{ex} as well as the *intrinsic* loss rate κ_{in}.

A.1. Linear detection of motion

First, we consider how motion is detected optically in such a setup, completely ignoring at first the effect of the light on the mechanical system. We make a few approximations for this particular analysis that are useful though not generally valid. First we assume that $x(t)$ is slow compared to optical bandwidth κ, or equivalently $\omega_m \ll \kappa$. Also we assume that the laser field is driving the photonic cavity on resonance so $\Delta = 0$, and that the optical decay rate is dominated by the out-coupling so $\kappa = \kappa_{ex}$. The output field is then given by $\alpha_{out} = \alpha_{in} + \sqrt{\kappa}\alpha$, which for oscillations that are small, i.e. when the laser-cavity detuning is being modulated by the motion within the linear region of the cavity phase response such that $Gx \ll \kappa$, can be solved to obtain a relation representing phase modulation of the output field: $\alpha_{out} = -\alpha_{in}(1 - i4Gx(t)/\kappa)$. This is a first order expansion of $e^{-i\phi(t)}$ where $\phi(t) = Gx(t)/\kappa$. An alternate way of writing this expression is in terms of the intracavity field $\alpha(t)$ which, neglecting the mechanical motion, is given by $\alpha = -2\alpha_{in}/\sqrt{\kappa}$.

The output field is then

$$\alpha_{\text{out}} = -\alpha_{\text{in}} - i\sqrt{\Gamma_{\text{meas}}}x/x_{\text{zp}} \tag{2}$$

with the measurement rate defined as

$$\Gamma_{\text{meas}} \equiv \frac{4G^2 x_{\text{zp}}^2 |\alpha|^2}{\kappa} = \frac{4g_0^2|\alpha|^2}{\kappa} \tag{3}$$

being the rate at which photons are scattered from the laser beam to sidebands due to motion of amplitude x_{zp} and an intracavity optical field intensity of $|\alpha|^2$. The subscript represents *zero-point*, assigned in anticipation of the quantum analysis below though for a classical description x_{zp} can be used to normalize the above expression without changing the physics. The measurement rate as defined here is central to understanding the operation of optomechanical systems in the linear regime and will be used throughout the text below often denoted alternatively as $\gamma_{\text{OM}} \equiv \Gamma_{\text{OM}} \equiv \Gamma_{\text{meas}}$. The zero-point coupling rate $g_0 = Gx_{\text{zp}}$ defined here is consistent with the main text.

A.2. Back-action on the mechanical mode

Now we consider how the motion of the mechanical system is modified due to interaction with the optical field resonating in the structure. In addition to equation 1, to understand the back-action arising from the interplay between the optical field and mechanical motion, we must consider the dynamics of the motional degree of freedom:

$$\ddot{x}(t) + \gamma_{\text{in}}\dot{x} + \omega_{\text{m}}^2 x = (F_{\text{BA}}(t) + F_{\text{input}}(t))/m_{\text{eff}}. \tag{4}$$

The left-hand side of the above equation is simply the equation of motion for a damped harmonic oscillator and takes into account the dynamics of the modal degree of freedom being considered. The right-hand side of the equation are the forcing terms: $F_{\text{BA}}(t)$ is the optical back-action, while $F_{\text{input}}(t)$ is an input force which we use to understand the linear response of the mechanical system. The back-action force is given by radiation pressure described via the Maxwell stress tensor, which is quadratic in the field or proportional to $|\alpha|^2$. By considering the total energy of the system (see section A.3), we find that $F_{\text{BA}}(t) = \hbar G|\alpha(t)|^2$. Equations 1 and 4 now describe the dynamics of the coupled system and can be solved to obtain the effects of back-action in the classical domain. In particular, we are primarily interested in the modification of the linear response of the mechanical system to an input force, i.e. changes to its damping rate and frequency. These changes come about from the mechanical motion x modifying the intracavity field $\alpha(t)$ which then applies a force back onto x which can be proportional, lagging, or leading, leading to a redefinition of the mechanical system's complex frequency. To calculate the laser power and frequency dependence of these modifications, we choose an operating point $(\bar{\alpha}, \bar{x})$ and linearize the equations of motion by taking to account only the dynamics of the fluctuations $\delta x(t)$ and $\delta\alpha(t)$. This gives us a set of three coupled linear differential equations

$$\begin{aligned}
\delta\ddot{x}(t) &= -\omega_{\text{m}}^2\delta x - \gamma_{\text{in}}\delta\dot{x} + \\
&\quad \hbar G(\bar{\alpha}^*\delta\alpha + \text{c.c.})/m_{\text{eff}} + F_{\text{input}}(t)/m_{\text{eff}} && (5)\\
\delta\dot{\alpha}(t) &= -(i\Delta + \kappa/2)\delta\alpha - iG\bar{\alpha}\delta x - \sqrt{\kappa_{\text{ex}}}\delta\alpha_{\text{in}} && (6)\\
\delta\dot{\alpha}^*(t) &= (i\Delta - \kappa/2)\delta\alpha^* + iG\bar{\alpha}^*\delta x - \sqrt{\kappa_{\text{ex}}}\delta\alpha_{\text{in}}^\star && (7)
\end{aligned}$$

which can be solved for input forces $F_{\text{input}}(t)$ taking $\delta\alpha_{\text{in}} = 0$ for now. Solving these equations in the Fourier domain, we obtain an expression for the small-signal response of the mechanical system to the input force in terms of a dispersion relation, $\delta x(\omega) \equiv \chi_x(\omega)F_{\text{input}}(\omega)$, with

$$\chi_x(\omega) = \frac{1}{m_{\text{eff}}(\omega_{\text{m}}^2 - \omega^2 - i\omega\gamma_{\text{in}} + \Sigma_{\text{opt}}(\omega))}, \tag{8}$$

where

$$\Sigma_{\text{opt}}(\omega) = -i\hbar G^2|\bar{\alpha}|^2\left(\chi_\alpha(\omega) - \chi_\alpha^*(-\omega)\right)/m_{\text{eff}} \tag{9}$$

and $\chi_\alpha(\omega) = (i(\Delta - \omega) + \kappa/2)^{-1}$ is the optical resonance response function. The expression in equation 8 represents the response of a damped mechanical resonance that is modified by a "self-energy" term, $\Sigma_{\text{opt}}(\omega)$, due to interaction with optical resonance. The real and imaginary parts of this self-energy cause an effective modification of the mechanical frequency and linewidth ω_{m} and γ_{in}. This shift in the complex frequency, often referred to as the "optical spring" and "optical damping/amplification" effects can be expressed succinctly in terms of $\Sigma_{\text{opt}}(\omega)$:

$$\gamma_{\text{OM}} \approx -\frac{\text{Im}\Sigma_{\text{opt}}(\omega_{\text{m}})}{\omega_{\text{m}}}, \text{ and} \tag{10}$$

$$\delta\omega_{\text{m}} \approx \frac{\text{Re}\Sigma_{\text{opt}}(\omega_{\text{m}})}{2\omega_{\text{m}}}. \tag{11}$$

These expression are good approximations in the weak-coupling regime ($g \ll \kappa + \gamma$). In the strong-coupling regime ($g \gg \kappa + \gamma$), the full frequency-dependence of the self-energy $\Sigma_{\text{opt}}(\omega)$ should be considered.

A common mode of operation of optomechanical systems that are sideband-resolved ($\omega_{\text{m}} \gg \kappa$) is to tune the laser approximately a mechanical frequency to the red side of the optical resonance so $\Delta = \omega_{\text{m}}$. In this case, the above relations lead us to $\gamma_{\text{OM}} \approx 2\hbar G^2|\bar{\alpha}|^2/(\kappa m_{\text{eff}}\omega_{\text{m}}) = 4G^2x_{\text{zp}}^2|\bar{\alpha}|^2/\kappa$ which is seen to be equal to the measurement rate calculated in equation 3, though that was for a different regime of operation. The equality of these two rates can be understood as such: with the red-detuned scheme of driving, all of the sideband scattering, which occurs at rate Γ_{meas}, causes up-shifting of the laser photons into the photonic mode, and thus effectively damps the mechanical resonator's motion. In the above we focused on the effect of the optomechanical interaction on the mechanical resonator's response function. However, there are equally important changes in the electromagnetic response. These effects, including Brillouin gain/loss and slow/fast light, can be derived similarly [2] (section B).

A.3. Understanding optomechanical coupling in a nanophotonic system

In the previous section we studied how optomechanical coupling can be used to detect mechanical motion and modify the linear response of a mechanical resonator. Here we will see how such an interaction comes about in a realistic nanophotonic system. Though a toy model with a one-dimensional scalar wave-equation and a simplified mass-spring system has long been used in studying optomechanical systems, obtaining a precise understanding of the coupling rates given nontrivial wavelength-scale optical and elastic mode profiles requires careful consideration of the fields and calculation of the interactions. The goal of this section is to show how we can obtain equations similar to equations 5-7 where $x(t)$ and $\alpha(t)$ now represent mode amplitudes for acoustic and optical excitations in a nanophotonic device.

We start by solving separately the dynamical equations for electromagnetics and elastodynamics which can be expressed as eigenvalue equations for the magnetic field $\mathbf{H}(\mathbf{r})$ and elastic displacement field $\mathbf{Q}(\mathbf{r})$ respectively:

$$\mathsf{L}\mathbf{h}_j = \omega_j^2 \mathbf{h}_j, \qquad \mathsf{L}(\cdot) = c^2 \mathrm{curl}\left[\frac{\epsilon_0}{\overline{\overline{\epsilon}}(\mathbf{r})}\mathrm{curl}\,(\cdot)\right]. \tag{12}$$

$$\begin{aligned}\omega_j^2 \mathbf{Q}_j(\mathbf{r}) &= \mathsf{L}\mathbf{Q}_j(\mathbf{r}) \\ &\mathsf{L}(\cdot) = -\frac{\lambda+\mu}{\rho}\nabla(\nabla\cdot(\cdot)) - \frac{\mu}{\rho}\nabla^2(\cdot).\end{aligned} \tag{13}$$

The set of solutions of these two equations are the normal electromagnetic and acoustic modes of the structure, and define the spectrum. Typically, a software package such as COMSOL is used to obtain these solutions in dielectric structures that don't permit analytic analysis. Valid solutions of the electromagnetic and elastic field in the structure can then be expressed as normal mode expansions

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_j \mathbf{e}_j(\mathbf{r}) a_j(t) + \text{h.c.}, \tag{14}$$

$$\hat{\mathbf{Q}}(\mathbf{r},t) = \sum_k \mathbf{Q}_k(\mathbf{r}) \hat{b}_k(t) + \text{h.c.} \tag{15}$$

with $\dot{a}_j(t) = -i\omega_j a_j(t)$, and $\dot{b}_k(t) = -i\omega_k \hat{b}_k(t)$. In defining the quantum field theory, we assign to each mode j a Hilbert space $\{|n\rangle_j, n = 0,1,2,3,..\}$ where $|n\rangle_j$ is the state representing n photons or phonons in the jth mode. Phonons and photons in each of these Hilbert spaces are then annihilated with the operators a_j and b_j respectively. The normalizations of $\mathbf{e}_j$ and $\mathbf{Q}_k$ in the equations above are then physically significant since, *e.g.*, the expectation value of $\hbar\omega_j a_j^\dagger a_j$ represents the energy stored in the jth mode of the electromagnetic field. We can use the classical expressions for energy in the fields to calculate the energy for a single photon/phonon above the vacuum state, $|\psi\rangle = |1\rangle_j$, which we will then set equal to $\hbar\omega_j$:

$$\begin{aligned}\hbar\omega_j \equiv U_{\text{mech}}^{|\psi\rangle} &= \langle\psi| \int \mathrm{d}\mathbf{r}\, \dot{\hat{\mathbf{Q}}}(\mathbf{r})\rho(\mathbf{r})\dot{\hat{\mathbf{Q}}}(\mathbf{r})|\psi\rangle \\ &\quad - \langle\text{vac}| \int \mathrm{d}\mathbf{r}\, \dot{\hat{\mathbf{Q}}}(\mathbf{r})\rho(\mathbf{r})\dot{\hat{\mathbf{Q}}}(\mathbf{r})|\text{vac}\rangle \\ &= 2\omega_j^2 \int \mathrm{d}\mathbf{r}\, \mathbf{Q}_j^*(\mathbf{r})\rho(\mathbf{r})\mathbf{Q}_j(\mathbf{r}) \\ &= 2m_{\text{eff}}\omega_j^2 \max[|\mathbf{Q}_j(\mathbf{r})|^2].\end{aligned} \tag{16}$$

where we've defined the effective mass and zero-point motion of the mode to be

$$m_{\text{eff},j} = \frac{\int \mathrm{d}\mathbf{r}\, \mathbf{Q}_j^*(\mathbf{r})\rho(\mathbf{r})\mathbf{Q}_j(\mathbf{r})}{\max[|\mathbf{Q}_j(\mathbf{r})|^2]}, \quad \text{and}$$

$$x_{\text{zpf},j} \equiv \max[|\mathbf{Q}_j(\mathbf{r})|] = \sqrt{\frac{\hbar}{2m_{\text{eff},j}\omega_j}}. \tag{17}$$

A similar consideration for the electromagnetic fields leads to

$$\begin{aligned}\hbar\omega_j \equiv U_{\text{em}} &= \langle\psi| \int \mathrm{d}\mathbf{r}\, \hat{\mathbf{E}}(\mathbf{r})\overline{\overline{\epsilon}}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r})|\psi\rangle - \\ &\quad \langle\text{vac}| \int \mathrm{d}\mathbf{r}\, \hat{\mathbf{E}}(\mathbf{r})\overline{\overline{\epsilon}}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r})|\text{vac}\rangle \\ &= 2\int \mathrm{d}\mathbf{r}\, \mathbf{e}_j^*(\mathbf{r})\overline{\overline{\epsilon}}(\mathbf{r})\mathbf{e}_j(\mathbf{r}) \\ &= 2V_{\text{eff}}\max[\mathbf{e}_j^*(\mathbf{r})\overline{\overline{\epsilon}}(\mathbf{r})\mathbf{e}_j(\mathbf{r})].\end{aligned} \tag{18}$$

with

$$V_{\text{eff},j} = \frac{\int \mathrm{d}\mathbf{r}\, \mathbf{e}_j^*(\mathbf{r})\overline{\overline{\epsilon}}(\mathbf{r})\mathbf{e}_j(\mathbf{r})}{\max[\mathbf{e}_j^*(\mathbf{r})\overline{\overline{\epsilon}}(\mathbf{r})\mathbf{e}_j(\mathbf{r})]}., \quad \text{and}$$

$$\max[|\mathbf{e}_j(\mathbf{r})|] = \sqrt{\frac{\hbar\omega_j}{2V_{\text{eff},j}\epsilon_{\text{diel}}}}. \tag{19}$$

Having defined the quantization of the fields and mode normalizations, we can now write a Hamiltonian for the optomechanical system,

$$\mathcal{H} = \underbrace{\hbar\sum_j \omega_j a_j^\dagger a_j}_{\mathcal{H}_{\text{o}}} + \underbrace{\hbar\sum_j \omega_j b_j^\dagger b_j}_{\mathcal{H}_{\text{m}}} + \mathcal{H}_{\text{int}} \tag{20}$$

that can capture the quantum dynamics. The challenge remains calculation of the optomechanical interaction term $\mathcal{H}_{\text{int}}$. We are interested in interactions of the type $a_j^\dagger a_k(b_l + b_l^\dagger)$ which are the simplest type that allow energy conservation, assuming that the photonic frequencies for mode j and k are many orders of magnitude larger than the mechanical frequency of mode l. For the case where $j = k$, we are considering a shift in the frequency of optical mode j due to a mechanical displacement in mode l. The relevant interaction energy or rate can be calculated using first-order cavity perturbation theory. In a dielectric structure characterized by $\overline{\overline{\epsilon}}_0(\mathbf{r})$, modifications due to deformations can be taken into account with the expression

$$\overline{\overline{\epsilon}}(\mathbf{r}) = \overline{\overline{\epsilon}}_0(\mathbf{r}) + \overline{\overline{\delta\epsilon}}(\mathbf{r}). \tag{21}$$

To first order, such a modification of the dielectric causes a shift in the optical resonance frequency of a mode with mode profile $\mathbf{e}(\mathbf{r})$ of

$$\omega^{(1)} = -\frac{\omega_0}{2}\frac{\langle\mathbf{e}|\overline{\overline{\delta\epsilon}}|\mathbf{e}\rangle}{\langle\mathbf{e}|\overline{\overline{\epsilon}}|\mathbf{e}\rangle}. \tag{22}$$

There are two ways that the dielectric constant changes due to this deformation. First, a deformation of the optical resonator affects the dielectric tensor at the *boundaries* between different materials. This is because the high-contrast step profile of $\overline{\overline{\epsilon}}(\mathbf{r})$ across a boundary is shifted by deformations of the structure. By relating a deformation to a change in the dielectric constant, we can use equation 22 to calculate the optomechanical coupling. Johnson has derived a useful expression [3] for this shift in frequency, which when adapted to optomechanics [4], gives a frequency shift per unit displacement of

$$g_{\text{OM,B}} = -\frac{\omega_0}{2}\frac{\int Q_n(\mathbf{r})(\Delta\overline{\overline{\epsilon}}|\mathbf{e}^{\parallel}|^2 - \Delta(\overline{\overline{\epsilon^{-1}}})|\mathbf{d}^{\perp}|^2)\mathrm{d}A}{\max(|\mathbf{Q}|)\int \overline{\overline{\epsilon}}(\mathbf{r})|\mathbf{e}(\mathbf{r})|^2\mathrm{d}^3\mathbf{r}} \tag{23}$$

for a mechanical vector displacement field $\mathbf{Q}(\mathbf{r})$ with component $Q_n(\mathbf{r})$ normal to the boundary. Secondly, a *photoelastic* contribution to the optomechanical coupling arises from local changes in the refractive index due to stress in the material induced by the mechanical deformation. For a particular displacement vector $\mathbf{Q}(\mathbf{r})$ corresponding strain tensor $S_{ij} = \frac{1}{2}\left(\partial_i Q_j + \partial_j Q_i\right)$, the dielectric perturbation is given by

$$\overline{\overline{\delta\epsilon}}(\mathbf{r}) = \overline{\overline{\epsilon}}\cdot\frac{\overline{\overline{p}}\cdot\overline{\overline{S}}}{\epsilon_0}\cdot\overline{\overline{\epsilon}}, \tag{24}$$

which reduces to $\delta\epsilon_{ij} = -\epsilon_0 n^4 p_{ijkl}S_{kl}$ for an isotropic medium [5, 6]. The fourth-rank tensor $\overline{\overline{p}}$ with components p_{ijkl} is called the photoelastic tensor and is a property of the material. The *roto-optic* effect, which captures permittivity changes induced by rotation, must be included as well in optically anisotropic materials [7–9]. Composite metamaterials may yield enhanced photoelasticity [9]. Often, when considering the symmetries in the atomic structure of the material, a reduced tensor is used with elements p_{ij}. The perturbation integral can then be used to calculate the shift in frequency per unit displacement:

$$g_{\mathrm{OM,PE}} = -\frac{\omega_0}{2} \frac{\int \mathbf{e} \cdot \overline{\overline{\delta\epsilon}} \cdot \mathbf{e}\, \mathrm{d}^3\mathbf{r}}{\max(|\mathbf{Q}|) \int \overline{\overline{\epsilon}}(\mathbf{r}) |\mathbf{e}(\mathbf{r})|^2 \mathrm{d}^3\mathbf{r}}. \tag{25}$$

These expressions give the boundary and photoelastic components for a shift in the optical cavity frequency per unit displacement of the maximum deflection point of a deformation profile $\mathbf{Q}(\mathbf{r})$. A natural unit for displacement is the zero-point fluctuation amplitude found by multiplying the expressions (23) and (25) by the zero-point fluctuation length $x_{\mathrm{zpf}} = \sqrt{\hbar/(2m_{\mathrm{eff}}\omega_{\mathrm{m}})}$ (see equation 17). The coupling rate is $g_0 = g_{0,\mathrm{Bnd}} + g_{0,\mathrm{PE}}$, and the corresponding optomechanical interaction Hamiltonian can be written as

$$\begin{aligned} \mathcal{H}_{\mathrm{int}} &= \hbar(g_{\mathrm{OM,PE}} + g_{\mathrm{OM,Bnd}}) x a^\dagger a \\ &= \hbar g_0 (b^\dagger + b) a^\dagger a. \end{aligned} \tag{26}$$

Recently, an open-source software tool called "NumBAT" [10] has been developed to compute the above overlap integrals for 2D-confined structures.

B. Waveguides: 2D-confined

A theory for 2D-confined waveguides with translational symmetry can be developed similarly to the previous section on 3D-confined cavities. We refer to [11–13] for a thorough treatment from first principles. Here, we focus on connecting the waveguides' dynamics to the cavities' dynamics described in the previous section. For a waveguide the translational symmetry implies that the photonic and phononic eigenproblems can be expressed in terms of wavevectors β and K, yielding as solution dispersion relations $\omega(\beta)$ and $\Omega(K)$. As shown in [11–13], the Hamiltonian of the waveguide is $\mathcal{H} = \mathcal{H}_{\mathrm{free}} + \mathcal{H}_{\mathrm{int}}$ with the free Hamiltonian given by

$$\mathcal{H}_{\mathrm{free}} = \int \mathrm{d}\beta\, \hbar\omega_\beta a_\beta^\dagger a_\beta + \int \mathrm{d}K\, \hbar\omega_K b_K^\dagger b_K \tag{27}$$

and the interaction Hamiltonian given by

$$\mathcal{H}_{\mathrm{int}} = \frac{\hbar}{\sqrt{2\pi}} \int\int \mathrm{d}\beta\, \mathrm{d}K \left(g_{\beta+K} a_\beta b_K + g_{\beta-K} a_\beta b_K^\dagger + \mathrm{h.c.} \right) \tag{28}$$

in the momentum-description where the three-wave mixing interaction rate $g_{\beta\pm K} = g_{0|\beta\pm K}\alpha^\dagger_{\beta\pm K}$ is proportional to the pump amplitude $\alpha_{\beta\pm K}$ of the mode with wavevector $\beta \pm K$. Here we assume phase-matching ($\Delta K = -\beta_{\mathrm{p}} + \beta \pm K = 0$ with β_{p} the pump wavevector) and usually consider α to represent a strong pump that can be treated classically.

Previous work [2] linked the waveguide's coupling rate $g_{0|\beta\pm K}$ and Brillouin gain coefficient $\mathcal{G}_{\mathrm{B}}$ to an optomechanical cavity's coupling rate g_0 via a mean-field transition performed on the photonic and phononic equations of motion both in the limit of large and small cavity finesse. Here, we derive the same connection but now via a mean-field transition in the large-finesse limit performed directly on the waveguide's Hamiltonian given by expression 28. We consider a cavity of roundtrip length L constructed from a waveguide described by equation 28 and focus on a triplet of two photonic and one phononic mode(s). The operator corresponding to the number of excitations in each of the modes can be expressed as

$$\begin{aligned} c^\dagger c &= \int_{k_c-\pi/L}^{k_c+\pi/L} \mathrm{d}k\, c_k^\dagger c_k \\ &= \frac{2\pi}{L} c_{k_c}^\dagger c_{k_c} \end{aligned} \tag{29}$$

with k_{c} a relevant center wavevector. Therefore, the cavity and waveguide operators are linked by

$$c = \sqrt{\frac{2\pi}{L}}\, c_{k_c} \tag{30}$$

with c either a photonic or phononic ladder operator. Considering the first term in $\mathcal{H}_{\mathrm{int}}$, we therefore have

$$\int\int \mathrm{d}\beta\, \mathrm{d}K\, g_{\beta+K} a_\beta b_K = \left(\frac{2\pi}{L}\right)^2 g_{\beta+K} a_\beta b_K \tag{31}$$

$$= \frac{2\pi}{L} g_{0|\beta+K} \alpha_{\beta+K} \delta a \delta b \tag{32}$$

$$= \frac{g_{0|\beta+K}}{\sqrt{L}} \alpha \delta a \delta b \tag{33}$$

$$= g_0 \alpha \delta a \delta b \tag{34}$$

with δa and δb the cavity's photonic and phononic ladder operators. Thus we obtain

$$g_0 = \frac{g_{0|\beta+K}}{\sqrt{L}} \tag{35}$$

as in the main text. This link between the coupling rates holds both for forward and backward as well as for intra- and intermodal scattering [2].

Next, we consider the waveguide's dynamics in slightly more detail. The dynamics of an optomechanical waveguide is usually considered after transforming from momentum- to real-space operators

$$c(z) = \int \frac{\mathrm{d}k}{2\pi} e^{-i(k-k_c)z} c_k \tag{36}$$

We are usually concerned with narrow bandwidths relative to the group-velocity-dispersion so the frequencies can be expanded to first-order as

$$\omega_k \approx \omega_{\mathrm{c}} + v_{\mathrm{g}}(k - k_{\mathrm{c}}) \tag{37}$$

A few manipulations [11] starting from equation 27 lead to

$$\mathcal{H}_{\mathrm{free}} = \hbar \int \mathrm{d}z \left[a^\dagger(z) \hat{\omega}_a a(z) + b^\dagger(z) \hat{\omega}_{\mathrm{m}} b(z) \right] \tag{38}$$

with $\hat{\omega}_k = \omega_{\mathrm{c}} - i v_{\mathrm{g}} \partial_z$ the real-space operator corresponding to the momentum-space dispersion relation ω_{k}. Higher-order expansions of the dispersion relation in equation 37 yield higher-order spatial derivatives the operator $\hat{\omega}_k$.

Further, dropping the second term in equation 28 the real-space interaction Hamiltonian becomes

$$\mathcal{H}_{\mathrm{int}} = \hbar \int \mathrm{d}z\, g_{0|\beta+K} \alpha^\dagger(z) a(z) b(z) + \mathrm{h.c.} \tag{39}$$

Further, the Heisenberg equations of motion $\dot{c}(z) = -\frac{i}{\hbar}[c(z), \mathcal{H}_{\mathrm{int}}]$ together with the equal-time commutator $[c(z), c^\dagger(z')] = \delta(z - z')$ yield

$$\begin{aligned} (\partial_t + v_\alpha \partial_z)\alpha &= -i\omega_\alpha \alpha - i g_{0|\beta+K} a\, b \\ (\partial_t + v_a \partial_z) a &= -i\omega_a a - i g_{0|\beta+K} \alpha\, b^\dagger \\ (\partial_t + v_{\mathrm{m}} \partial_z) b &= -i\omega_{\mathrm{m}} b - i g_{0|\beta+K} \alpha\, a^\dagger \end{aligned} \tag{40}$$

These equations describe the spatiotemporal evolution of the three interacting fields in absence of dissipation and phase-mismatch. Intrinsic propagation losses can be included via a dissipative term, e.g.

$$(\partial_t + v_{\mathrm{m}}\partial_z)b = -i\omega_{\mathrm{m}}b - \gamma b - ig_{0|\beta+K}\alpha a^\dagger \tag{41}$$

for the phononic field with $\gamma = v_{\mathrm{m}}\alpha_{\mathrm{m}}$ with $L_{\mathrm{m}} = \alpha_{\mathrm{m}}^{-1}$ the phononic decay length. Similarly, a non-zero phase-mismatch $\Delta K \neq 0$ effectively reduces the interaction rate. In particular, dropping the second term for simplicity the momentum-space interaction Hamiltonian 28 becomes

$$\mathcal{H}_{\mathrm{int}} = \frac{\hbar}{\sqrt{2\pi}} \int\int\int \mathrm{d}\beta\,\mathrm{d}K\,\mathrm{d}\beta_{\mathrm{p}} \left(g_{0|\beta_{\mathrm{p}}} \alpha^\dagger_{\beta_{\mathrm{p}}} a_\beta b_K L \,\mathrm{sinc}\left(\frac{\Delta K L}{2}\right) + \mathrm{h.c.}\right) \tag{42}$$

with L the waveguide length. Thus the finite length weakens the wavevector-selectivity, generating interactions between a larger set of modes. The strongest interactions are obtained between modes for which $\Delta K = 0$. This corresponds to the real-space Hamiltonian

$$\mathcal{H}_{\mathrm{int}} = \hbar \int \mathrm{d}z\, g_{0|\beta+K}\alpha^\dagger(z)a(z)b(z)\,e^{i\Delta K_{\mathrm{c}}z} + \mathrm{h.c.} \tag{43}$$

with $\Delta K_{\mathrm{c}} = -\beta_{\mathrm{pc}} + \beta_{\mathrm{c}} + K_{\mathrm{c}}$ the phase-mismatch between the center wavevectors of the photonic and phononic fields. This suppresses the interaction in the equations of motion via a rotating term, for instance

$$(\partial_t + v_{\mathrm{m}}\partial_z)b = -i\omega_{\mathrm{m}}b - ig_{0|\beta_{\mathrm{p}}}\alpha a^\dagger\, e^{-i\Delta K_{\mathrm{c}}z} \tag{44}$$

The range of spatiotemporal effects described by these and extended versions of these equations of motion are considered in detail in amongst others [2, 11, 12, 14–19].

B. SINGLE-PHOTON NONLINEARITY

In this section we give a derivation for the relations given at the end of the perspective on single-photon nonlinearities in the main text. We consider a 2D-confined waveguide of length L and inject a photon flux $\langle\Phi\rangle = v_{\mathrm{g}}/L$ with v_{g} the optical group velocity that corresponds to one photon on average in the waveguide. This photon excites the mechanical system with a displacement x_1, which in turn yields a phase-shift ϑ_{wg} on a second probe photon. The phase-shift can be expressed as

$$\vartheta_{\mathrm{wg}} = k_0(\partial_x n_{\mathrm{eff}})x_1 L \tag{45}$$

with k_0 the vacuum optical wavevector and $\partial_x n_{\mathrm{eff}}$ the sensitivity of the effective optical index to mechanical motion x_1. Assuming a static displacement, we have $x_1 = \langle F\rangle/k_{\mathrm{eff}}$ with $\langle F\rangle$ the force exerted by the first photon and k_{eff} the effective mechanical stiffness per unit length. In addition, from power-conservation it can be shown that $\langle F\rangle = \frac{1}{c}\partial_x n_{\mathrm{eff}}(\hbar\omega)\langle\Phi\rangle$ [20]. Substitution leads to

$$\vartheta_{\mathrm{wg}} = \frac{\hbar\omega^2}{k_{\mathrm{eff}}}\left(\frac{1}{c}\partial_x n_{\mathrm{eff}}\right)^2 v_{\mathrm{g}} \tag{46}$$

Since the Brillouin gain coefficient can be expressed as [21]

$$\mathcal{G}_{\mathrm{B}} = 2\omega\frac{Q_{\mathrm{m}}}{k_{\mathrm{eff}}}\left(\frac{1}{c}\partial_x n_{\mathrm{eff}}\right)^2 \tag{47}$$

this yields

$$\vartheta_{\mathrm{wg}} = \frac{\hbar\omega}{2}\frac{\mathcal{G}_{\mathrm{B}}}{Q_{\mathrm{m}}}v_{\mathrm{g}} \tag{48}$$

Making use of the connection [2] discussed in the main text

$$\mathcal{G}_{\mathrm{B}} = \frac{4g^2_{0|\beta+K}}{v_{\mathrm{p}}v_{\mathrm{s}}(\hbar\omega)\gamma} \tag{49}$$

leads to

$$\vartheta_{\mathrm{wg}} = \frac{2g^2_{0|\beta+K}}{v_{\mathrm{g}}\omega_{\mathrm{m}}} \tag{50}$$

This is the single-photon cross-phase shift for a statically driven mechanical waveguide. When the mechanics is driven with a detuning $\Delta\Omega > \gamma$ close to the resonance, one must replace $x_1 \to (\omega_{\mathrm{m}}/(2\Delta\Omega))x_1$ so the cross-Kerr phase shift increases to

$$\vartheta_{\mathrm{wg}} = \frac{g^2_{0|\beta+K}}{v_{\mathrm{g}}\Delta\Omega} \tag{51}$$

which is in agreement with more rigorous analysis [22]. This phase-shift is independent of the waveguide length L since the photon flux and the optical forces are inversely proportional to length L when there is on average a single photon in the waveguide.

Next, we link the waveguide single-photon phase-shift ϑ_{wg} to the cavity single-photon phase shift

$$\vartheta_{\mathrm{cav}} = \frac{2g_0^2}{\kappa\Delta\Omega} \tag{52}$$

derived in the main paper. Making use of

$$g_0 = \frac{g_{0|\beta+K}}{\sqrt{L}} \tag{53}$$

the cavity phase-shift ϑ_{cav} can be expressed as

$$\vartheta_{\mathrm{cav}} = \frac{2g^2_{0|\beta+K}}{L_{\mathrm{rt}}\kappa\Delta\Omega} \tag{54}$$

with L_{rt} the cavity roundtrip length. The cavity finesse is $\mathcal{F} = 2\pi/(\kappa T_{\mathrm{rt}})$ with $T_{\mathrm{rt}} = L_{\mathrm{rt}}/v_{\mathrm{g}}$ the cavity roundtrip time so we obtain

$$\vartheta_{\mathrm{cav}} = \frac{\mathcal{F}}{\pi}\vartheta_{\mathrm{wg}} \tag{55}$$

The expression for the cavity phase-shift assumed critical coupling and a small phase shift given by $\vartheta_{\mathrm{cav}} = 2\Delta/\kappa$ with $\Delta = \vartheta_{\mathrm{wg}}/T_{\mathrm{rt}}$ the mechanically-induced detuning from the cavity resonance.

CONTRIBUTIONS

R.V.L. organized and wrote much of the manuscript along with A.S.N. The section on microwave signal processing was written by D.V.T. All authors read, discussed and gave critical feedback on the paper. The authors are grateful to two anonymous referees for helpful feedback.

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