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Analytical description of a luminescent solar concentrator device: supplementary material

ILYA SYCHUGOV

Department of Applied Physics, School of Engineering Sciences, KTH-Royal Institute of Technology, Stockholm, Sweden
ilyas@kth.se

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This document provides supplementary information to "Analytical description of a luminescent solar concentrator device," https://doi.org/10.1364/OPTICA.6.001046. Here detailed analytical derivations are provided together with additional verification simulations.

Section S1. Derivation of the optical path length distribution in a slab.

We are interested in the probability density for a photon to travel distance r to the edge of a rectangular slab for an isotropic point-like source randomly placed inside it.

A) **2D case in-plane (XY plane).**

Consider edge element of a length dy. The fraction of the isotropic emission from a point source at distance r into the edge element dy:

$$f = \frac{d\varphi}{2\pi}$$

From the yellow triangle:

$$dy \cdot sin(\pi - \theta) = 2 \cdot (r - dy \cdot cos(\pi - \theta)) \cdot sin(d\varphi/2)$$

For small φ one can approximate $\sin(d\varphi/2) \approx d\varphi/2$. For small dy one can simplify $(r - dy \cdot \cos(\pi - \theta)) \approx r$. Then

$$dy \cdot sin(\theta) \approx r \cdot d\varphi$$

Then the fraction f for a single source becomes:

$$f = \frac{dy \cdot sin(\theta)}{2\pi r}$$

Elementary length ds of the arc with radius r contributing to the signal for the angle θ (blue segment):

$$ds = \frac{d\theta}{2\pi} \cdot 2\pi r = r d\theta$$

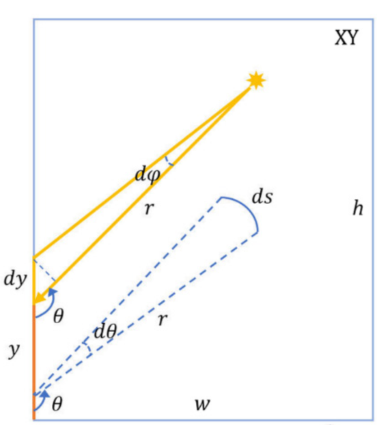
If the arc is limited by angles $\theta_{1,2}$ the total fraction of photons from the arc of a length s, arriving to the element dy (summing up signal from all the sources at a distance r):

$$F = \int_0^s f ds = \frac{dy}{2\pi} \int_{\theta_1}^{\theta_2} sin(\theta)\, d\theta = \frac{dy}{2\pi}(cos(\theta_1) - cos(\theta_2))$$

Finally, for the total fraction of photons reaching the edge (length h) after travelling distance r one should integrate over the whole edge length:

$$p(r) = \frac{1}{2\pi} \int_0^h (cos(\theta_1) - cos(\theta_2))\, dy$$

which, after normalization, represents the probability density function. Since limiting angles $\theta_{1,2}$ vary depending on the geometry and the exact position of dy several cases should be considered. For certainty a rectangular with a width smaller than the height ($w < h$) is taken into account.



Scheme S1. Notations used in the derivation of the section S1A.

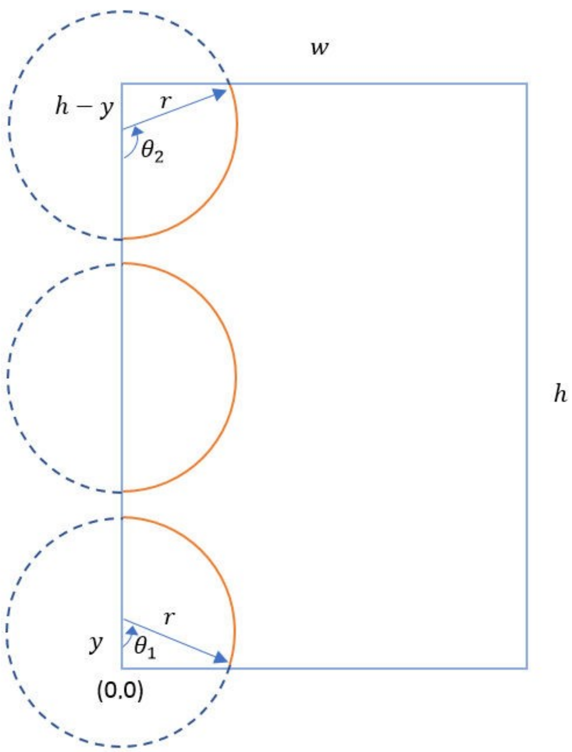
A1) For $r < \frac{h}{2}$:

Three different cases can be considered:

1. For $0 < y < r$ the arc is from θ_1 to π, where $\cos(\theta_1) = \frac{y}{r}$
2. For $r < y < h - r$ the arc is from 0 to π
3. For $h - r < y < h$ the arc is from 0 to θ_2, where $\cos(\pi - \theta_2) = \frac{h-y}{r}$, $\cos(\theta_2) = -\frac{h-y}{r}$

And the total number of photons can be calculated by integrating respective parts:

$$\boldsymbol{p_{left1}(r) = \frac{1}{2\pi}\left(\int_0^r \left(\frac{y}{r} + 1\right) dy + \int_r^{h-r} 2dy + \int_{h-r}^{h} \left(1 + \frac{h-y}{r}\right) dy\right) = \frac{2h - r}{2\pi}}$$



Scheme S2. Notations used in the derivation of the section S1A1.

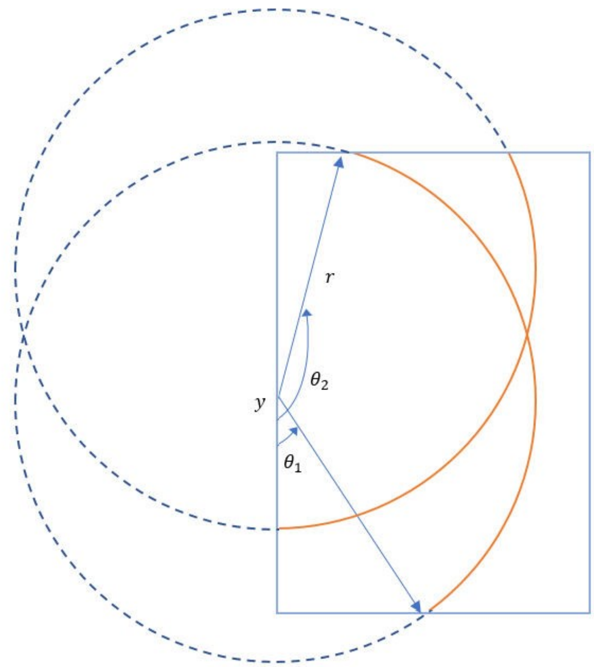
A2) For $\frac{h}{2} < r < w$

Again three different cases can be considered:

1. For $0 < y < h - r$ the arc is from θ_1 to π
2. For $h - r < y < r$ the arc is from θ_1 to θ_2
3. For $r < y < h$ the arc is from 0 to θ_2

$$\boldsymbol{p_{left2}(r) = \frac{1}{2\pi}\left(\int_0^{h-r} \left(\frac{y}{r} + 1\right) dy + \int_{h-r}^{r} \left(\frac{y}{r} + \frac{h-y}{r}\right) dy + \int_r^{h} \left(1 + \frac{h-y}{r}\right) dy\right) = \frac{2h - r}{2\pi}}$$

The same result as for the case above.



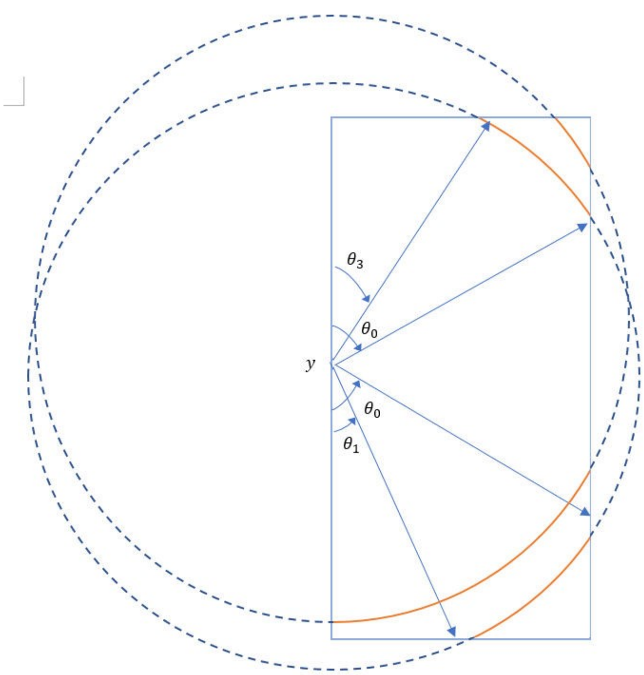
Scheme S3. Notations used in the derivation of the section S1A2.

A3) For $w < r < h$

Two cases are:

1. For $0 < y < h - r$ the arc is from 0 to θ_0, where $\sin(\theta_0) = \frac{w}{r}$, $\theta_0 = \arcsin\left(\frac{w}{r}\right)$
2. For $h - r < y < h - \sqrt{r^2 - w^2}$ the arc is from θ_3 to θ_0, where $\theta_3 = \pi - \theta_2 = \arccos\left(\frac{h-y}{r}\right)$

These two arcs repeat themselves from the other side, so their respective contributions should be multiplied by 2.



Scheme S4. Notations used in the derivation of the section S1A3.

$$p_{left3}(r) = 2\int_0^{h-r}\left(1 - cos\left(arcsin\left(\frac{w}{r}\right)\right)\right)dy + 2\int_{h-r}^{h-\sqrt{r^2-w^2}}\left(\frac{h-y}{r} - cos\left(arcsin\left(\frac{w}{r}\right)\right)\right)dy = \frac{2hr - w^2 - 2h\sqrt{r^2-w^2}}{2\pi r}$$

A4) For $h < r < \sqrt{h^2+w^2}$

Two cases are:

1. For $0 < y < h - \sqrt{r^2-w^2}$ the arc is from θ_3 to θ_0
2. For $\sqrt{r^2-w^2} < y < h$ the arc is from θ_1 to θ_0

$$p_{left4}(r) = \int_0^{h-\sqrt{r^2-w^2}}\left(\frac{h-y}{r} - cos\left(arcsin\left(\frac{w}{r}\right)\right)\right)dy + \int_{\sqrt{r^2-w^2}}^{h}\left(\frac{y}{r} - cos\left(arcsin\left(\frac{w}{r}\right)\right)\right)dy = \frac{r^2 - w^2 - 2h\sqrt{r^2-w^2} + h^2}{2\pi r}$$

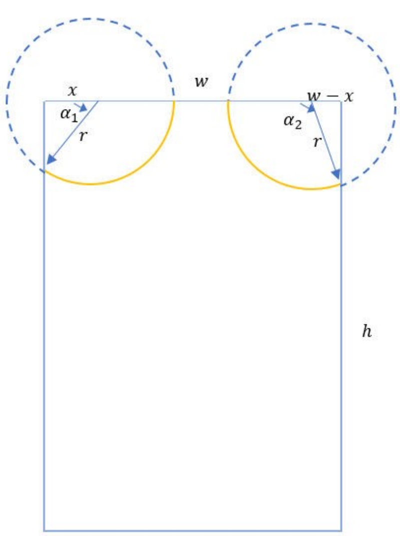
One can repeat such derivations for the top edge:

A5) For $r < \frac{w}{2}$

Similar to the first case for the left facet:

1. For $0 < x < r$ the arc is from α_1 to π, where $\alpha_1 = \arccos\left(\frac{x}{r}\right)$
2. For $r < x < h - r$ the arc is from 0 to π
3. For $h - r < x < h$ the arc is from 0 to α_2, where $\alpha_2 = \arccos\left(-\frac{w-x}{r}\right)$

$$p_{top1}(r) = \frac{2w - r}{2\pi}$$



Scheme S5. Notations used in the derivations of the section S1A5.

A6) For $\frac{w}{2} < r < w$

1. For $0 < x < w - r$ countable arc is the same as above: from α_1 to π
2. For $w - r < x < r$ it is from α_1 to α_2
3. For $r < x < w$ it is the same as above: from 0 to α_2

$$p_{top2}(r) = \frac{2w - r}{2\pi}$$

A7) For $w < r < h$ (same range as for left facet)

Only one case needs to be considered:

1. For $0 < x < w$ the countable arc is from α_1 to α_2

$$p_{top3}(r) = \int_0^{w}\left(\frac{x}{r} + \frac{w-x}{r}\right)dx = \frac{w^2}{2\pi r}$$

A8) For $h < r < \sqrt{h^2+w^2}$

1. For $0 < x < w - \sqrt{r^2-h^2}$ Countable arc is from α_3 to α_0, where $\alpha_0 = \arcsin\left(\frac{h}{r}\right), \alpha_3 = \pi - \alpha_2 = \arccos\left(\frac{w-x}{r}\right)$
2. For $\sqrt{r^2-h^2} < x < w$ Countable arc is from α_1 to α_0

$$p_{top4}(r) = \int_0^{w-\sqrt{r^2-h^2}}\left(\frac{w-x}{r} - cos\left(arcsin\left(\frac{h}{r}\right)\right)\right)dx + \int_{\sqrt{r^2-h^2}}^{w}\left(\frac{x}{r} - cos\left(arcsin\left(\frac{h}{r}\right)\right)\right)dx = \frac{r^2 - h^2 - 2w\sqrt{r^2-h^2} + w^2}{2\pi r}$$

Then for the total perimeter of the rectangular (2 left and 2 top edges) a piecewise and continuous function $p(r)$ can be defined as (Figure 1):

$$\begin{cases} p_1(r) = \dfrac{2w + 2h - 2r}{\pi}, & 0 < r < w \\ p_2(r) = \dfrac{2h(r - \sqrt{r^2-w^2})}{\pi r}, & w < r < h \\ p_3(r) = \dfrac{2r^2 - 2h\sqrt{r^2-w^2} - 2w\sqrt{r^2-h^2}}{\pi r}, & h < r < d \end{cases}$$

Normalization coefficient is just the area of the rectangular hw, as would be expected:

$$\int_0^{\sqrt{h^2+w^2}} p(r)dr = \int_0^{w} p_1(r)dr + \int_w^{h} p_2(r)dr + \int_h^{\sqrt{h^2+w^2}} p_3(r)dr =$$

$$= \frac{w(w+2h)}{\pi} + \frac{h}{\pi}\left(2h + (\pi-2)w - 2\sqrt{h^2-w^2} - 2w \cdot arctan\left(\frac{w}{\sqrt{h^2-w^2}}\right)\right) + \frac{1}{\pi}\left(2h\sqrt{h^2-w^2} + \pi wh - w^2 - 2h^2 - 2\,hw \cdot arctan\left(\frac{w}{h}\right) - 2hw \cdot arctan\left(\frac{h\sqrt{h^2-w^2}-w^2}{w\sqrt{h^2-w^2}+wh}\right)\right) =$$

$$= \frac{2hw}{\pi}\left(\pi - arctan\left(\frac{w}{h}\right) - arctan\left(\frac{w}{\sqrt{h^2-w^2}}\right) - arctan\left(\frac{h\sqrt{h^2-w^2}-w^2}{w\sqrt{h^2-w^2}+wh}\right)\right) = hw$$

Average value of the distribution:

$$\rho =< r >= \frac{\int_0^{\sqrt{h^2+w^2}} r \cdot p(r)dr}{\int_0^{\sqrt{h^2+w^2}} p(r)dr}$$

First moment (nominator):

$$\frac{1}{3\pi}\left(h^3 + w^3 - (h^2+w^2)^{\frac{3}{2}} + 3hw^2\, ln\left(\frac{\sqrt{h^2+w^2}+h}{w}\right) + 3wh^2\, ln\left(\frac{\sqrt{h^2+w^2}+w}{h}\right)\right)$$

Then the average photon optical path from an isotropic emitter randomly placed in a rectangular:

$$\rho = \frac{1}{3\pi hw}\left(h^3 + w^3 - (h^2+w^2)^{\frac{3}{2}} + 3hw^2\, ln\left(\frac{\sqrt{h^2+w^2}+h}{w}\right) + 3wh^2\, ln\left(\frac{\sqrt{h^2+w^2}+w}{h}\right)\right)$$

For a simple case of a square slab ($h = w = a$):

$$\rho_{sq} = \frac{2\left(1 + 3\,ln\left(1+\sqrt{2}\right) - \sqrt{2}\right)}{3\pi} a \approx 0.47a$$

B) 2D case out-of-plane (XZ plane)

Emitted light from an isotropic emitter reflects many times from the media boundary due to the total internal reflection when the light is emitted outside the escape cone. Individual optical path length between reflections for the light emitted below critical angle θ_{c1} ($\sin(\theta_{c1}) = 1/n$, $\theta_{c1} \approx 42^{o}$ for $n = 1.5$ of glass or polymers):

$$l_1 = \frac{\Delta}{cos(\theta)}$$

Total optical path in this plane for N bounces until reaching the edge

$$l = N \cdot l_1 = \frac{r}{l_1 \cdot sin(\theta)} \cdot l_1 = \frac{r}{sin(\theta)}$$

where $\theta_{c1} < \theta < \theta_{c2} = \pi - \theta_{c1}$. So l does not deviate much from the distance to the edge r, and is in the range $r < l < nr = \frac{3}{2}r$ (for glass or polymers), depending on the angle θ.

Probability density function for the light emitted from an isotropic emitter is constant $0 < \theta < 2\pi$:

$$g(\theta) = \frac{dP}{d\theta} = \frac{1}{2\pi}$$

Changing variable to the optical path length l for a given parameter r

$$q_r(l) = \frac{dP}{dl} = \frac{dP}{d\theta} \cdot \left|\frac{d\theta}{dl}\right| = g(\theta) \cdot \left|\frac{d\theta}{dl}\right|$$

Where

$$\theta = arcsin\left(\frac{r}{l}\right)$$

Therefore

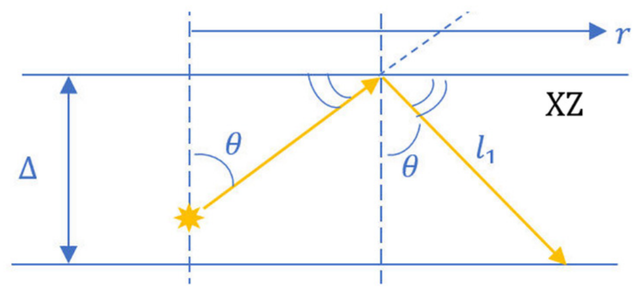
$$\frac{d\theta}{dl} = -\frac{r}{l\sqrt{l^2-r^2}}$$

Then we obtain

$$q_r(l) = \frac{1}{2\pi} \cdot \frac{r}{l\sqrt{l^2-r^2}}$$

An average value of the optical path from the distribution $q_r(l)$ is close to r:

$$< l >= \frac{\int_r^{3r/2} q_r(l') \cdot l' dl'}{\int_r^{3r/2} q_r(l')dl'} = \frac{2r\left(ln\left(3+\sqrt{5}\right) - ln(2)\right)}{\pi - 2arctan(2/\sqrt{5})} \cdot r = k \cdot r \approx 1.144 \cdot r$$



Scheme S6. Notations used in the derivations of the section S1B.

C) 3D case

The distance r from the derived distribution above $q_r(l)$ is not a constant, but has a probability density distribution $p(r)$, where the probability of having $r = r'$ is $p(r')dr'$ for a properly normalized probability density function. That corresponds to the distribution of the optical path lengths:

$$q(l) = \frac{1}{2\pi l}\int_{\frac{2}{3}l}^{l} \frac{p(r')r'}{\sqrt{l^2-r'^2}} dr'$$

Integration limits reflect the fact that only individual distributions with $\frac{2}{3}l < r' < l$ will contribute to the total probability density at the point l. Exact analytical solution is possible to obtain through special functions (complete and incomplete elliptic integrals). Using the following notations:

$$\eta = \frac{\sqrt{l^2 - w^2}}{l}, \varkappa = \sqrt{\frac{4l^2 - 9w^2}{4l^2 - 4w^2}},$$

$$\gamma = \frac{\sqrt{l^2 - h^2}}{l}, \chi = \frac{\sqrt{h^2 + w^2}}{l}$$

One can find that:

For $0 < l < w$

$$q_1(l) = \frac{1}{2\pi l}\int_{\frac{2}{3}l}^{l} \frac{p_1(r')r'}{\sqrt{l^2 - r'^2}} dr'$$

$$q_1(l) = \frac{(12h + 12w - 4l)\sqrt{5} - 9l\left(\pi - 2\,arcsin\left(\frac{2}{3}\right)\right)}{36\pi^2} \approx 0.076(h + w) - 0.068 \cdot l$$

If $w < \frac{2}{3}h$ then for $w < l < \frac{3}{2}w$ (otherwise for $w < l < h$)

$$q_2(l) = \frac{1}{2\pi l}\int_{\frac{2}{3}l}^{w} \frac{p_1(r')r'}{\sqrt{l^2 - r'^2}} dr' + \frac{1}{2\pi l}\int_{w}^{l} \frac{p_2(r')r'}{\sqrt{l^2 - r'^2}} dr'$$

$$q_2(l) = \frac{1}{18\pi^2}\Bigg((6h + 6w - 2l)\sqrt{5} - 9l\left(arcsin\left(\frac{w}{l}\right) - arcsin\left(\frac{2}{3}\right)\right) - 9wk + 18h\left(\frac{w^2}{l^2}K(\eta) - E(\eta)\right)\Bigg)$$

where K, E are complete elliptic integrals of the first and second kind respectively.

If $w < \frac{2}{3}h$ then for $\frac{3}{2}w < l < h$

$$q_3(l) = \frac{1}{2\pi l}\int_{\frac{2}{3}l}^{l} \frac{p_2(r')r'}{\sqrt{l^2 - r'^2}} dr'$$

$$q_3(l) = \frac{1}{\pi^2}\Bigg(\frac{h\sqrt{5}}{3} - hE(\eta) + h\frac{w^2}{l^2}\left(K(\eta) - F(\varkappa,\eta) + \Pi(\varkappa,\eta^2,\eta)\right)\Bigg)$$

where F, Π are incomplete elliptical integrals of the first and third kind respectively.

If $w < \frac{2}{3}h$ then for $h < l < \sqrt{h^2 + w^2}$

$$q_4(l) = \frac{1}{2\pi l}\int_{\frac{2}{3}l}^{h} \frac{p_2(r')r'}{\sqrt{l^2 - r'^2}} dr' + \frac{1}{2\pi l}\int_{h}^{l} \frac{p_3(r')r'}{\sqrt{l^2 - r'^2}} dr'$$

$$q_4(l) = \frac{1}{\pi^2}\Bigg(\frac{h\sqrt{5}}{3} - \frac{h}{2l}\sqrt{l^2 - h^2} + \frac{\pi l}{4} + h\frac{w^2}{l^2}\left(\Pi(\varkappa,\eta^2,\eta) - F(\varkappa,\eta) + K(\eta)\right) - hE(\eta) - wE(\gamma) - \frac{l}{2}arcsin\left(\frac{h}{l}\right) + h^2\frac{w}{l^2}K(\gamma)\Bigg)$$

If $w < \frac{2}{3}h$ then for $\sqrt{h^2 + w^2} < l < \frac{3}{2}h$

$$q_5(l) = \frac{1}{2\pi l}\int_{\frac{2}{3}l}^{h} \frac{p_2(r')r'}{\sqrt{l^2 - r'^2}} dr' + \frac{1}{2\pi l}\int_{h}^{\sqrt{h^2+w^2}} \frac{p_3(r')r'}{\sqrt{l^2 - r'^2}} dr'$$

$$q_5(l) = \frac{1}{\pi^2}\Bigg(arcsin(\chi)\frac{l}{2} - arcsin\left(\frac{h}{l}\right)\frac{l}{2} + h\frac{w^2}{l^2}\left(F\left(\frac{h}{\eta\chi l},\eta\right) - \Pi\left(\frac{h}{\eta\chi l},\eta^2,\eta\right) - F(\varkappa,\eta) + \Pi(\varkappa,\eta^2,\eta)\right) + h^2\frac{w}{l^2}\left(F\left(\frac{w}{\eta\chi l},\gamma\right) - \Pi\left(\frac{w}{\eta\chi l},\gamma^2,\gamma\right)\right) - \frac{\chi}{2}\sqrt{l^2 - h^2 - w^2} - \frac{h}{2l}\sqrt{l^2 - h^2}\Bigg)$$

To verify these formulas several millions of path lengths were numerically calculated for a point with a varying location inside a 3D slab with given side lengths. Resulting distributions (dots) indeed converge to the analytical expressions presented here (blue and red lines).

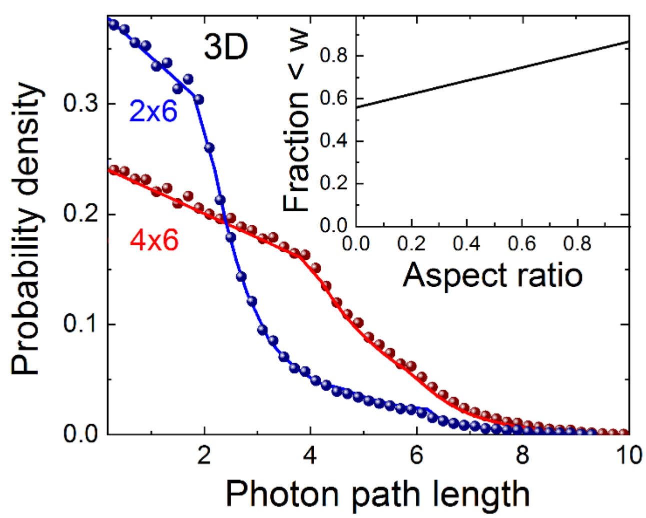


Figure S1. Probability density function distributions for a 3D slab with dimension 2x6 (blue) and 4x6 (red) units. Points are counted by simulating about a million paths from an isotropic emitter, and solid lines are analytical solutions from above. Inset shows fraction of optical paths below the width w of a rectangular (aspect ratio $\beta = w/h$).

Section S2. Approximate solution for the optical path distribution

A) Derivation of the approximate solution

The exact solution presented above is not very convenient to work with, so an approximate analytical solution would be easier to use instead. It can be obtained based on the fact that $q_r(l)$ varies only marginally, being chiefly close to r. So, as a first approximation, one can substitute distribution $q_r(l)$ by its average value $l \approx < l >= k \cdot r$ and to rely solely on the obtained 2D distribution $p(r)$. So the approximate analytical distribution for the optical path length distribution in 3D can be written as

$$\boldsymbol{q(l) \approx p(l/k)}$$

It appears to be a very good approximation for different aspect ratio geometries (Figure 1, inset). The meaning of the coefficient $k \approx 1.14$ can be then interpreted as a correction for 3D geometry from a 2D case. So the final solution $q'(l)$ becomes:

$$\begin{cases} \dfrac{2w+2h-2l/k}{\pi}, & 0<l<kw \\ \dfrac{2h(l-\sqrt{l^2-(kw)^2})}{\pi l}, & kw<l<kh \\ \dfrac{2l^2/k-2h\sqrt{l^2-(kw)^2}-2w\sqrt{l^2-(kh)^2}}{\pi l}, & kh<l<kd \end{cases}$$

Using normalization coefficient:

$$\int_0^{k\sqrt{h^2+w^2}} q'(l)dl = k \cdot \int_0^{\sqrt{h^2+w^2}} p(r)dr = khw$$

A properly normalized 3D probability density function $q(l)$ then becomes:

$$\begin{cases} \dfrac{2w+2h-2l/k}{\pi hwk}, & 0<l<kw \\ \dfrac{2l-2\sqrt{l^2-(kw)^2}}{\pi lwk}, & kw<l<kh \\ \dfrac{2l^2/k-2h\sqrt{l^2-(kw)^2}-2w\sqrt{l^2-(kh)^2}}{\pi lhwk}, & kh<l<kd \end{cases}$$

B) Probability of the optical path to be shorter than the rectangular width

The probability for an optical path to be shorter than the rectangular width w (aspect ratio $\beta = \frac{w}{h} \leq 1$):

$$P_w = \int_0^w q(l)dl = \frac{2}{k\pi}\left(\beta + 1 - \frac{\beta}{2k}\right) \approx 0.31\beta + 0.56$$

It is shown in the inset of Figure S1 as a function of β. So most of the photon path distribution lies below the shortest side of the rectangular. Even for a very large 1:5 ratio it is > 60% probability, reaching ~ 85% for the squared shape. For the "golden ratio" $\frac{2}{1+\sqrt{5}} \approx 0.62$ it is 75%. So for most practical applications it is possible to say that the rectangular width mainly limits optical path of photons in a 3D slab.

Section S3. Effect of matrix absorption

The probability of having optical path $l = l'$ is $q(l')dl'$ for a properly normalized probability density function $q(l)$. Then

$$f(\alpha) = \int_0^{lmax} q(l') \cdot exp(-\alpha l')dl'$$

Which essentially shows the fraction of photons reaching the edge for given h and w (diagonal $d = \sqrt{h^2+w^2}$) of a rectangular, where $l_{max} = kd$. Calculating dimensionless $f(\alpha)$ using obtain normalized distribution $q(l)$ yields:

$$f_1(\alpha) = \int_0^{kw} q_1(l') \cdot exp(-\alpha l')dl' = \frac{2}{\alpha^2 hw\pi k^2}\left((h+w)\alpha k - 1 + e^{-kw\alpha}(1 - kh\alpha)\right)$$

$$f_2(\alpha) = \int_{kw}^{kh} q_2(l') \cdot exp(-\alpha l')\,dl' = \frac{2}{\alpha w\pi k}\left(e^{-kw\alpha} - e^{-kh\alpha}\right) - \frac{2}{w\pi k}\int_{kw}^{kh} \frac{\sqrt{l^2-(kw)^2}}{l} exp(-\alpha l)dl$$

$$f_3(\alpha) = \int_{kh}^{kd} q_3(l') \cdot exp(-\alpha l')\,dl' = \frac{2}{\alpha^2 hw\pi k^2}\left(\alpha hk \cdot e^{-\alpha hk} - d\alpha k \cdot e^{-\alpha dk} + e^{-\alpha hk} - e^{-\alpha dk}\right) - \frac{2}{w\pi k}\int_{kh}^{kd} \frac{\sqrt{l^2-(kw)^2}}{l} exp(-\alpha l)\,dl - \frac{2}{h\pi k}\int_{kh}^{kd} \frac{\sqrt{l^2-(kh)^2}}{l} exp(-\alpha l)dl$$

$$f(\alpha) = f_1(\alpha) + f_2(\alpha) + f_3(\alpha)$$

$$f(\alpha) = \frac{2}{\alpha^2 hw\pi k^2}\left((h+w)\alpha k - (kd\alpha + 1) \cdot e^{-kd\alpha} + e^{-kh\alpha} + e^{-kw\alpha} - 1\right) - \mathfrak{I}_1 - \mathfrak{I}_2$$

where two integrals are:

$$\mathfrak{I}_1 = \frac{2}{w\pi k}\int_{kw}^{kd} \frac{\sqrt{l^2-(kw)^2}}{l} exp(-\alpha l)\,dl,$$

$$\mathfrak{I}_2 = \frac{2}{h\pi k}\int_{kh}^{kd} \frac{\sqrt{l^2-(kh)^2}}{l} exp(-\alpha l)dl$$

Section S4. Fraction of light emitted to the waveguiding mode

The emitted light from a fluorophore (quantum dot, organic dye, etc.) in a polymer/glass slab will experience total internal reflection for angles at the air interface larger than a critical angle α_c. In the most common case for a glass or a polymer: $n = 1.5$, $n_{air} = 1$ and the critical angle α_c:

$$sin(\alpha_c) = \frac{1}{n}$$

i.e. $\alpha_c \approx 42°$. Thus, for the emitter in a rectangular slab with six facets there are six cones with the angle $2\alpha_c$, where the emitted light can escape. Solid angle of the cone (surface of a spherical cap) for a unity radius sphere is

$$S_1 = 2\pi(1 - cos\,\alpha_c)$$

So the fraction of the emitted light through one facet is:

$$\delta_1 = \frac{S_1}{S} = \frac{2\pi(1 - \cos\alpha_c)}{4\pi} = \frac{\left(1 - \frac{\sqrt{n^2-1}}{n}\right)}{2}$$

Considering only emitted light through top and bottom facets as losses the total useful fraction of the emission is then ($n = 1.5$):

$$\delta = 1 - 2\delta_1 = \frac{\sqrt{n^2-1}}{n} \approx 75\%$$

Section S5. Effect of scattering by fluorophores

If the total loss is governed by the scattering instead (absorption-free matrix and re-absorption free fluorophore) then the optical efficiency can be also evaluated from the optical path length distribution. Let the linear scattering coefficient be $\alpha_{sc}[1/cm]$. Probability for the photon to travel optical path l' before reaching the edge is $q(l')dl'$. Probability of not being scattered within distance l' is $\exp(-\alpha_{sc}l')$. These photons will contribute to the total optical efficiency similarly to the absorption case above:

$$\chi_0(\alpha_{sc}) = \int_0^{lmax} q(l') \cdot exp(-\alpha_{sc}l')\, dl' = f(\alpha_{sc})$$

In addition, there will be photons, which underwent scattering into the waveguiding mode. Probability of being scattered within distance l' is $1 - \exp(-\alpha_{sc}l')$. If δ is a fraction of waveguided light after a scattering event (δ=75% for n=1.5) the probability to reach the edge after one scattering event is:

$$\chi_1(\alpha_{sc}) = \delta \cdot \int_0^{lmax} q(l') \cdot (1 - exp(-\alpha_{sc}l'))dl' \cdot \int_0^{lmax} q(l') \cdot exp(-\alpha_{sc}l')\, dl'$$

$$\chi_1(\alpha_{sc}) = \delta \cdot (1 - f(\alpha_{sc})) \cdot f(\alpha_{sc})$$

A Markov process is considered, where there is no memory in the system. The total probability for a photon to reach the edge becomes then a geometrical series (sum of probabilities for no scattering, one scattering, two scattering events, etc.):

$$\chi(\alpha_{sc}) = \sum_{i=0}^{\infty} \chi_i = f(\alpha_{sc})\left[1 + \delta \cdot \left(1 - f(\alpha_{sc})\right) + \left(\delta \cdot \left(1 - f(\alpha_{sc})\right)\right)^2 + \ldots\right] = \frac{f(\alpha_{sc})}{1 - \delta \cdot \left(1 - f(\alpha_{sc})\right)}$$

Section S6. Effect of several loss mechanisms present simultaneously

Now consider two processes taking place simultaneously: scattering and matrix absorption. First, photons experiencing no scattering and no absorption will contribute to the total signal:

$$\psi_0(\alpha_{sc}, \alpha) = \int_0^{lmax} q(l') \cdot \exp(-\alpha_{sc}l') \cdot \exp(-\alpha l')\, dl' = f(\alpha_{sc} + \alpha)$$

Then photons after one scattering event and without subsequent scattering and absorption. While every scattering event sets back to zero the travelled distance for scattering, the optical path for absorption continues. So the exact history of scattering becomes important. To take into account this fact one can introduce a probability *density* to scatter at a point l' (in the absence of other processes):

$$p_{sc}(l') = \alpha_{sc} \cdot exp(-\alpha_{sc}l')$$

which is a properly normalized probability density function. Then in the system where scattering and absorption coexist the probability density to scatter at a point l' without being absorbed before is:

$$p_{sc}(l') \int_{l'}^{\infty} p_{ab}(x)dx$$

where a similar notation of the probability density p_{ab} is introduced for the pure absorption process. Additional conditions of no subsequent scattering and absorption can be added as:

$$p_{sc}(l') \int_{l'}^{\infty} p_{ab}(x)dx \cdot \delta \cdot exp(-\alpha_{sc}l_2) \cdot exp(-\alpha l_2)$$

where l_2 is a photon path taken to reach the device edge after the scattering event. If l' varies in between $(0; l_1)$ the integrated *probability* becomes:

$$\delta\, exp(-(\alpha_{sc} + \alpha)l_2) \int_0^{l_1} \alpha_{sc}\, exp(-\alpha_{sc}l')\, exp(-\alpha l')\, dl' = exp(-(\alpha_{sc} + \alpha)l_2) \frac{\delta\alpha_{sc}}{\alpha_{sc} + \alpha}[1 - exp(-(\alpha_{sc} + \alpha)l_1)]$$

Finally taking into account probability to have photon path l_1 as $q(l_1)dl_1$ and l_2 as $q(l_2)dl_2$ (again Markov process without memory in the system considered) one obtains after integration from zero to l_{max} for both path stretches $l_{1,2}$ the input from the photons experienced one scattering event:

$$\psi_1(\alpha_{sc}, \alpha) = f(\alpha_{sc} + \alpha) \cdot \delta\alpha_{sc} \frac{[1 - f(\alpha_{sc} + \alpha)]}{\alpha_{sc} + \alpha}$$

Continuing in the same manner for two scattering events without subsequent scattering and absorption:

$$\psi_2(\alpha_{sc}, \alpha) = f(\alpha_{sc} + \alpha) \cdot (\delta\alpha_{sc})^2 \left(\frac{[1 - f(\alpha_{sc} + \alpha)]}{\alpha_{sc} + \alpha}\right)^2$$

So the resulting probability can be again represented through geometrical series:

$$\psi(\alpha_{sc}, \alpha) = \sum_{i=0}^{\infty} \psi_i(\alpha_{sc}, \alpha) = \frac{f(\alpha_{sc} + \alpha)}{1 - \delta \cdot \frac{\alpha_{sc}}{\alpha_{sc} + \alpha} \cdot [1 - f(\alpha_{sc} + \alpha)]}$$

This formula turns into the expression for scattering only scenario for a non-absorbing matrix ($\alpha = 0$). A similar result can be derived for the case of re-absorption instead of scattering:

$$\boldsymbol{\phi}(\boldsymbol{\alpha}_{re},\boldsymbol{\alpha}) = \frac{\boldsymbol{f}(\boldsymbol{\alpha}_{re}+\boldsymbol{\alpha})}{\mathbf{1}-\boldsymbol{\delta}\cdot\boldsymbol{QY}\cdot\dfrac{\boldsymbol{\alpha}_{re}}{\boldsymbol{\alpha}_{re}+\boldsymbol{\alpha}}\cdot[\mathbf{1}-\boldsymbol{f}(\boldsymbol{\alpha}_{re}+\boldsymbol{\alpha})]}$$

When re-absorption and scattering both exist in the system one can show in a similar manner as above:

$$\boldsymbol{\varphi}(\boldsymbol{\alpha}_{sc},\boldsymbol{\alpha}_{re}) = \frac{\boldsymbol{f}(\boldsymbol{\alpha}_{sc}+\boldsymbol{\alpha}_{re})}{\mathbf{1}-\dfrac{\boldsymbol{\delta}\cdot\boldsymbol{\alpha}_{sc}+\boldsymbol{\delta}\cdot\boldsymbol{QY}\cdot\boldsymbol{\alpha}_{re}}{\boldsymbol{\alpha}_{sc}+\boldsymbol{\alpha}_{re}}\big(\mathbf{1}-\boldsymbol{f}(\boldsymbol{\alpha}_{sc}+\boldsymbol{\alpha}_{re})\big)}$$

A general solution for the optical efficiency $g(\alpha_{sc},\alpha_{re},\alpha)$, following derivations above, is:

$$\boldsymbol{g} = \frac{\boldsymbol{f}(\boldsymbol{\alpha}_{sc}+\boldsymbol{\alpha}_{re}+\boldsymbol{\alpha})}{\mathbf{1}-\dfrac{\boldsymbol{\delta}\cdot\boldsymbol{\alpha}_{sc}+\boldsymbol{\delta}\cdot\boldsymbol{QY}\cdot\boldsymbol{\alpha}_{re}}{\boldsymbol{\alpha}_{sc}+\boldsymbol{\alpha}_{re}+\boldsymbol{\alpha}}\big(\mathbf{1}-\boldsymbol{f}(\boldsymbol{\alpha}_{sc}+\boldsymbol{\alpha}_{re}+\boldsymbol{\alpha})\big)}$$