Supplemental Document

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Nonlocal magnon entanglement generation in coupled hybrid cavity systems: supplement

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In this supplemental document, we provide detailed calculations of entanglement dynamics, optimal fiber coupling strength, and the treatment of non-Markovian open system dynamics.

1. STATE AND ENTANGLEMENT DYNAMICS IN THE CLOSED SYSTEM CASE

For the unitary evolution under the Hamiltonian Eq. (1) in the main text, first note that the Hamiltonian conserves the total number of excitations, which enables us to focus on the single-excitation subspace. In the single-excitation subspace $\{|100,000\rangle, |010,000\rangle \ldots |000,001\rangle\}$, where the basis of the system is $|c_1m_1q_1, c_2m_2q_2\rangle$ with c_i denotes cavity $i = 1,2$ and the SQ (magnon) in cavity i is denoted as $q_i(m_i)$. The Hamiltonian can be expressed in this basis as

$$H = \begin{pmatrix} \omega_c & g_m & g_q & J & 0 & 0 \\ g_m & \omega_m & 0 & 0 & 0 & 0 \\ g_q & 0 & \omega_q & 0 & 0 & 0 \\ J & 0 & 0 & \omega_c & g_m & g_q \\ 0 & 0 & 0 & g_m & \omega_m & 0 \\ 0 & 0 & 0 & g_q & 0 & \omega_q \end{pmatrix}. \tag{S1}$$

In the resonant case $\omega_c = \omega_m = \omega_q = \omega$, ω is a trivial diagonal constant and can therefore be dropped. The initial state is taken to have one excitation in the qubit 1, i.e., $|\psi(0)\rangle = |001,000\rangle$. The state at time t is then given by

$$|\psi(t)\rangle = \begin{pmatrix} -\frac{2ig_q\cos\left(\frac{Jt}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\Omega} \\[2ex] \frac{g_mg_q\left(J\sin\left(\frac{Jt}{2}\right)\sin\left(\frac{\Omega t}{2}\right)+\Omega\cos\left(\frac{Jt}{2}\right)\cos\left(\frac{\Omega t}{2}\right)-\Omega\right)}{\Omega\left(g_m^2+g_q^2\right)} \\[2ex] \frac{g_q^2\left(\frac{J\sin\left(\frac{Jt}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\Omega}+\cos\left(\frac{Jt}{2}\right)\cos\left(\frac{\Omega t}{2}\right)\right)+g_m^2}{g_m^2+g_q^2} \\[2ex] -\frac{2g_q\sin\left(\frac{Jt}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\Omega} \\[2ex] -\frac{ig_mg_q\left(\sin\left(\frac{Jt}{2}\right)\cos\left(\frac{\Omega t}{2}\right)-\frac{J\cos\left(\frac{Jt}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\Omega}\right)}{g_m^2+g_q^2} \\[2ex] -\frac{ig_q^2\left(\sin\left(\frac{Jt}{2}\right)\cos\left(\frac{\Omega t}{2}\right)-\frac{J\cos\left(\frac{Jt}{2}\right)\sin\left(\frac{\Omega t}{2}\right)}{\Omega}\right)}{g_m^2+g_q^2} \end{pmatrix}, \tag{S2}$$

where $\Omega = \sqrt{4\left(g_m^2+g_q^2\right)+J^2}$.

The concurrence between the magnons (qubits/cavities/qubit1-magnon2) (C_m, C_q, C_c, C_{qm}) can be easily obtained. The reduced density operator for the two magnons (in computational basis

$|11\rangle, |10\rangle, |01\rangle, |00\rangle$) is given as

$$\rho_{mm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & w & 0 \\ 0 & w^* & y & 0 \\ 0 & 0 & 0 & z \end{pmatrix}, \tag{S3}$$

where

$$x = \frac{g_m^2 g_q^2 \left[J \sin\left(\frac{Jt}{2}\right) \sin\left(\frac{\Omega t}{2}\right) + \Omega \cos\left(\frac{Jt}{2}\right) \cos\left(\frac{\Omega t}{2}\right) - \Omega\right]^2}{\Omega^2 \left(g_m^2 + g_q^2\right)^2}, \tag{S4}$$

$$y = \frac{g_m^2 g_q^2 \left[\Omega \sin\left(\frac{Jt}{2}\right) \cos\left(\frac{\Omega t}{2}\right) - J \cos\left(\frac{Jt}{2}\right) \sin\left(\frac{\Omega t}{2}\right)\right]^2}{\Omega^2 \left(g_m^2 + g_q^2\right)^2}, \tag{S5}$$

$$z = 1 - x - y, \tag{S6}$$

$$\begin{aligned} w = {} & \frac{i g_m^2 g_q^2 \left[\Omega \sin\left(\frac{Jt}{2}\right) \cos\left(\frac{\Omega t}{2}\right) - J \cos\left(\frac{Jt}{2}\right) \sin\left(\frac{\Omega t}{2}\right)\right]}{\Omega^2 \left(g_m^2 + g_q^2\right)^2} \\ & \times \left[J \sin\left(\frac{Jt}{2}\right) \sin\left(\frac{\Omega t}{2}\right) + \Omega \cos\left(\frac{Jt}{2}\right) \cos\left(\frac{\Omega t}{2}\right) - \Omega\right]. \end{aligned} \tag{S7}$$

This is a mixed state in X form. Its concurrence can be obtained directly as

$$\begin{aligned} C_m = {} & 2 \frac{g_m^2 g_q^2}{G_0 \left(g_m^2 + g_q^2\right)^2} \left| \left[\sqrt{G_0} \cos \frac{\sqrt{G_0} t}{2} \sin \frac{Jt}{2} - J \cos \frac{Jt}{2} \sin \frac{\sqrt{G_0} t}{2}\right] \right| \\ & \times \left| \left[J \sin \frac{Jt}{2} \sin \frac{\sqrt{G_0} t}{2} + \sqrt{G_0} \left(\cos \frac{\sqrt{G_0} t}{2} \cos \frac{Jt}{2} - 1\right)\right] \right|, \end{aligned} \tag{S8}$$

where $G_0 = \Omega^2 = 4\left(g_m^2 + g_q^2\right) + J^2$.

2. OPTIMAL TIME AND COUPLING FOR REACHING PEAK ENTANGLEMENT

In this section we provide the details of achieving peak entanglement with optimal time and fiber coupling through the analysis of the time dependent entanglement given as C_m in Eq. (S8). We denote $C_m = |\widetilde{C}_m|$, and proceed to find the extrema of $\widetilde{C}_m$ through the partial derivative

$$\partial_t \widetilde{C}_m \propto \sin \frac{\sqrt{G_0} t}{2} \left[J \sin\left(\frac{\sqrt{G_0} t}{2}\right) \cos(Jt) + \sqrt{G_0} \left(\sin\left(\frac{Jt}{2}\right) - \cos\left(\frac{\sqrt{G_0} t}{2}\right) \sin(Jt)\right)\right]. \tag{S9}$$

The global minimum/maximum may occur along t where $\sin \frac{\sqrt{G_0} t}{2} = 0$. The maximum can be reached when $t = 2n\pi/\sqrt{G_0}$, subject to an optimal J. To find this optimal coupling value, we observe that at such times, the concurrence is given by

$$C_m|_t = 2 \left| \frac{g_m^2 g_q^2 \left(1 - (-1)^n \cos \frac{Jn\pi}{\sqrt{G_0}}\right) \sin \frac{Jn\pi}{\sqrt{G_0}}}{\left(g_m^2 + g_q^2\right)^2} \right| \tag{S10}$$

$$= \frac{2 g_m^2 g_q^2}{\left(g_m^2 + g_q^2\right)^2} \left| \left[1 - (-1)^n \cos x\right] \sin x \right|, \tag{S11}$$

where $x = \frac{Jn\pi}{\sqrt{G_0}}$. For odd n, the maximum of $|(1 + \cos x) \sin x| = 3\sqrt{3}/4$ is reached when $\cos x = 1/2$; for even n, the maximum of $|(1 - \cos x) \sin x| = 3\sqrt{3}/4$ is reached when $\cos x = -1/2$. We'll get the same result if we calculate $\partial_J[C_m|_t] = 0$, which gives $\cos(x) = -(-1)^n/2$ (corresponds to max) and $\cos(x) = (-1)^n$ (corresponds to $C_m = 0$). Therefore, the optimal J is given by:

$$J = 2(6m+1)\sqrt{\frac{g_m^2+g_q^2}{9n^2-(6m+1)^2}}, \quad 6m+1 < 3n, \text{odd } n \tag{S12}$$

$$J = 2(6m+5)\sqrt{\frac{g_m^2+g_q^2}{9n^2-(6m+5)^2}}, \quad 2m+2 \leq n, \text{odd } n \tag{S13}$$

or

$$J = 4(3m+1)\sqrt{\frac{g_m^2+g_q^2}{9n^2-4(3m+1)^2}}, \quad 2m+1 \leq n, \text{even } n \tag{S14}$$

$$J = 4(3m+2)\sqrt{\frac{g_m^2+g_q^2}{9n^2-4(3m+2)^2}}, \quad 6m+4 < 3n, \text{even } n \tag{S15}$$

where m is a non-negative integer. Labeling the cases of Eq. (S12) through Eq. (S15) as $c1 \ldots c4$, the corresponding optimal time $t = 2n\pi/\sqrt{G_0}$, we have the optimal times given by

$$t_{c1} = \frac{1}{3}\pi\sqrt{\frac{9n^2-(6m+1)^2}{g_m^2+g_q^2}}, \quad 6m+1 < 3n, \text{odd } n \tag{S16}$$

$$t_{c2} = \frac{1}{3}\pi\sqrt{\frac{9n^2-(6m+5)^2}{g_m^2+g_q^2}}, \quad 2m+2 \leq n, \text{odd } n \tag{S17}$$

$$t_{c3} = \frac{1}{3}\pi\sqrt{\frac{9n^2-4(3m+1)^2}{g_m^2+g_q^2}} \quad 2m+1 \leq n, \text{even } n \tag{S18}$$

$$t_{c4} = \frac{1}{3}\pi\sqrt{\frac{9n^2-4(3m+2)^2}{g_m^2+g_q^2}} \quad 6m+4 < 3n, \text{even } n \tag{S19}$$

It can then be readily shown, for case 1, we write odd $n = 2*u+1$, where u is a non-negative integer,

$$t_{c1} = \frac{1}{3}\pi\sqrt{\frac{9(2u+1)^2-(6m+1)^2}{g_m^2+g_q^2}}, \quad 3m < 3u+1. \tag{S20}$$

Since $u > m - 1/3$ and u is an integer, we have $u \geq m$, so

$$\begin{aligned} t_{c1} &\geq \frac{1}{3}\pi\sqrt{\frac{9(2m+1)^2-(6m+1)^2}{g_m^2+g_q^2}} \\ &\geq \frac{2}{3}\pi\sqrt{\frac{6m+2}{g_m^2+g_q^2}} \\ &\geq \frac{2\sqrt{2}\pi}{3\sqrt{g_m^2+g_q^2}} \end{aligned} \tag{S21}$$

For case 2, since n is odd, one has $n \geq 2m+2 \Rightarrow n \geq 2m+3$. Combining with the fact that $2m+2$ is even, one can obtain

$$\begin{aligned} t_{c2} &\geq \frac{1}{3}\pi\sqrt{\frac{9(2m+3)^2-(6m+5)^2}{g_m^2+g_q^2}} \\ &\geq \frac{2}{3}\sqrt{2}\pi\sqrt{\frac{6m+7}{g_m^2+g_q^2}} \\ &\geq \frac{2\sqrt{14}\pi}{3\sqrt{g_m^2+g_q^2}}. \end{aligned} \tag{S22}$$

Likewise, for case 3, since n is even, one has $n \geq 2m+1 \Rightarrow n \geq 2m+2$. Combining with the fact that $2m+1$ is odd, one can achieve

$$
\begin{aligned}
t_{c3} &\geq \frac{1}{3}\pi\sqrt{\frac{9(2m+2)^2-4(3m+1)^2}{g_m^2+g_q^2}} \\
&\geq \frac{4}{3}\pi\sqrt{\frac{3m+2}{g_m^2+g_q^2}} \\
&\geq \frac{4\sqrt{2}\pi}{3\sqrt{g_m^2+g_q^2}}.
\end{aligned}
\tag{S23}
$$

For case 4, since n is even, we write $n = 2u$, where u is a positive integer, thus

$$
t_{c4} = \frac{2}{3}\pi\sqrt{\frac{9u^2-(3m+2)^2}{g_m^2+g_q^2}}, \quad 3m+2 < 3u.
\tag{S24}
$$

Since $u > m + 2/3$ and u is an integer, we have $u \geq m+1$,

$$
\begin{aligned}
t_{c4} &\geq \frac{2}{3}\pi\sqrt{\frac{9(m+1)^2-(3m+2)^2}{g_m^2+g_q^2}} \\
&\geq \frac{2}{3}\pi\sqrt{\frac{6m+5}{g_m^2+g_q^2}} \\
&\geq \frac{2\sqrt{5}\pi}{3\sqrt{g_m^2+g_q^2}}
\end{aligned}
\tag{S25}
$$

Thus, the *global* minimum time is reached for case 1 when $n = 1$, $m = 0$, and is given by

$$
t = \frac{2\sqrt{2}\pi}{3\sqrt{g_m^2+g_q^2}},
\tag{S26}
$$

with a corresponding optimal coupling

$$
J = \frac{\sqrt{g_m^2+g_q^2}}{\sqrt{2}},
\tag{S27}
$$

and the maximum concurrence is

$$
C_m|_{\max} = \frac{3\sqrt{3}g_m^2 g_q^2}{2\left(g_m^2+g_q^2\right)^2}.
\tag{S28}
$$

Along with the result $J_f \approx \sqrt{8\pi c\Gamma_c/L}$ in Ref. [1], these expressions above allow us to estimate the length of the fiber for optimal entanglement generation. For example, with couplings $g_m/2\pi = 21$ MHz, $g_q/2\pi = 30$ MHz, and leaking rate $\Gamma_c/2\pi = 1.8$ MHz, the distance can be approximately 32 meters, generating a peak concurrence of around 0.57 in about 41 nanoseconds (corresponding to $n = 3$, $m = 0$). Under the same parameters, if one is willing to wait about 259 nanoseconds (corresponding to $n = 19$, $m = 0$), the corresponding distance is about 1307 meters, with the same peak concurrence.

For varying $g_{m(q)}$, $C_m|_{\max}$ has a maximum value of $\frac{3\sqrt{3}}{8}$, reached when $|g_m| = |g_q|$. It's noteworthy that in this case, $g_m = g_q = J$, and the eigen-energy of the system Hamiltonian become evenly-spaced (with a two-fold degeneracy at zero energy)

$$
E_{1\ldots6} = \{-2J, -J, 0, 0, J, 2J\}.
\tag{S29}
$$

Similar features can be found for qubit-qubit and qubit-magnon entanglement.

3. OPEN SYSTEM DYNAMICS

For the non-Markovian open system dynamics, we consider the system embedded in an infinite-mode bosonic bath, where the total Hamiltonian is given by

$$H = H_s + H_{\text{bath}} + H_{\text{int}}, \tag{S30}$$

where $H_{\text{bath}} = \sum_{j=1,2}\sum_k \omega_k \bar{b}^\dagger_{j,k}\bar{b}_{j,k}$, $H_{\text{int}} = \sum_{j=1,2}\sum_k g_k L_j \bar{b}^\dagger_{j,k} + g_k^* L_j^\dagger \bar{b}_{j,k}$. H_{int} describes the interaction between the cavity system and bath, ω_k is the frequency of the k-th bath mode, g_k is the corresponding coupling strength, L_j $(j = 1, 2)$ are the cavity-bath coupling operators and $\bar{b}_{j,k}$ denotes the k-th mode annihilation operator of bath j. Here, we consider leaky cavities with the coupling operators $L_j = \Gamma_c a_j$. The quantum state diffusion (QSD) equation which describes a set of stochastic trajectories [2] is used to calculate the non-Markovian dynamics. Projecting the two baths onto the Bargmann coherent state $|z^{(j)}\rangle = e^{z_{k,j}\bar{b}_{j,k}}|0\rangle$, $j = 1, 2$, we have the QSD equation

$$\partial_t|\psi_t(z_1,z_2)\rangle = \left[-iH + L_1 z_1^*(t) + L_2 z_2^*(t) - L_1^\dagger \bar{O}_1(t) - L_2^\dagger \bar{O}_2(t)\right]|\psi_t(z_1,z_2)\rangle, \tag{S31}$$

where $z_j^*(t) \equiv -i\sum_k g_k z_{k,j}^* e^{i\omega_k t}$, and the leading order noise-free $O_{1,2}(t,s)$ satisfies

$$\partial_t O_i(t,s) = \left[-iH - L_1^\dagger \bar{O}_1(t) - L_2^\dagger \bar{O}_2(t), O_i(t,s)\right], \tag{S32}$$

where $\bar{O}_i(t) = \int_0^t \alpha(t,s) O_i(t,s) ds$, $i = 1, 2$. The reduced density operator for the system is given by an ensemble average of the trajectories, $\rho_s(t) = \mathcal{M}[|\psi_t(z_1,z_2)\rangle\langle\psi_t(z_1,z_2)|]$. This leads to the non-Markovian master equation

$$\partial_t \rho_s(t) = [-iH, \rho_s(t)] + \left[L_1, \rho_s(t)\bar{O}_1^\dagger\right] + \left[L_2, \rho_s(t)\bar{O}_2^\dagger\right] - \left[L_1^\dagger, \bar{O}_1\rho_s(t)\right] - \left[L_2^\dagger, \bar{O}_2\rho_s(t)\right]. \tag{S33}$$

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