Supplemental Document

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Light propagation in a three-dimensional Rydberg gas with a nonlocal optical response: supplement

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Appendix: Susceptibility in k space

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To derive the susceptibilities $\tilde{\chi}_l$ (Eq.(16)) and $\tilde{\chi}_n$ (Eq. (17)) in k space,applied the following rules of the Fourier transform (here, we definition: $\mathcal{F}[f(x)] = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$):
(i) Convolution theorem: For $f(x) = f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(x')f_2(x-x')dx'$, where $*$ denotes the convolution operation, then:

$$\tilde{f}(k) = \sqrt{2\pi}\tilde{f}_1(k)\cdot\tilde{f}_1(k)$$

and

$$\mathcal{F}[f_1 f_2] = \frac{1}{\sqrt{2\pi}}\tilde{f}_1(k) * \tilde{f}_2(k);$$

(ii) Translation/ time-Shifting: For any real number x_0, we have

$$\mathcal{F}[f(x-x_0)] = e^{-ix_0k}\tilde{f}(k);$$

(iii) Modulation/ frequency shifting: For any real number k_0, if $f'(x) = e^{ixk_0}f(x)$, then

$$\mathcal{F}[f'(x)] = \tilde{f}(k-k_0);$$

(iv) Time scaling: For a nonzero real number a,

$$\mathcal{F}[f(ax)] = \frac{1}{|a|}\tilde{f}\left(\frac{k}{a}\right).$$

We now perform a Fourier transform on the susceptibility. For the local term, we have

$$\begin{aligned}
&\mathcal{F}[\chi_l(\mathbf{r})E(\mathbf{r})] = \mathcal{F}\left[g\int_{-\infty}^{\infty}\alpha_l(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')d\mathbf{r}'E(\mathbf{r})\right]\\
=\ & \frac{1}{\sqrt{2\pi}}\mathcal{F}\left[g\int_{-\infty}^{\infty}\alpha_l(\mathbf{r}-\mathbf{r}'))n(\mathbf{r}')d\mathbf{r}'\right] * \mathcal{F}\left[E(\mathbf{r})\right]\\
=\ & \frac{1}{\sqrt{2\pi}}\left(\mathcal{F}\left[\int_{-\infty}^{\infty}A_0 n(\mathbf{r}')d\mathbf{r}'\right] + \mathcal{F}\left[\int_{-\infty}^{\infty}\alpha^+(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')d\mathbf{r}'\right]\right) * \mathcal{F}\left[E(\mathbf{r})\right]\\
=\ & \left(\frac{1}{\sqrt{2\pi}}\mathcal{F}\left[A_0\right] + \mathcal{F}\left[\alpha^+(\mathbf{r})\right]\mathcal{F}\left[n(\mathbf{r})\right]\right) * \mathcal{F}\left[E(\mathbf{r})\right]\\
=\ & \left(A_0\delta(\mathbf{k}) + \sqrt{2\pi}n\tilde{\alpha}^+(\mathbf{k})\delta(\mathbf{k})\right) * \tilde{E}(\mathbf{k})\\
=\ & \left(A_0 + \sqrt{2\pi}n\tilde{\alpha}^+(0)\right)\tilde{E}(\mathbf{k})\\
=\ & \tilde{\chi}_l\tilde{E}(\mathbf{k}).
\end{aligned} \tag{1}$$

Here $A_0 = 2g\delta/(\Omega_c^2 - 2i\Gamma\delta)$, $\alpha^{\pm}(\mathbf{r}-\mathbf{r}') = \frac{g\Omega_c^2}{2i\Gamma(2i\Gamma\delta-\Omega_c^2)}(A_+(\mathbf{r}-\mathbf{r}') \pm A_-(\mathbf{r}-\mathbf{r}'))$,

$$\begin{aligned}
&\mathcal{F}\left[A_{\pm}(\mathbf{r}-\mathbf{r}')\right] = \mathcal{F}\left[\frac{1}{1+(r/R_b^{\pm})^6}\right]\\
=\ & \frac{1}{(2\pi)^{3/2}}\int\int\int\frac{1}{1+(r/R_b^{\pm})^6}e^{-i(k_x x+k_y y+k_z z)}dxdydz
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty \int_0^{2\pi} \int_0^\pi \frac{1}{1+(r/R_b^\pm)^6} e^{-ikr\cos\theta} r^2 \sin\theta d\theta d\varphi dr \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{1+(r/R_b^\pm)^6} r^2 \int_0^\pi e^{-ikr\cos\theta} \sin\theta d\theta dr \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{1+(r/R_b^\pm)^6} \frac{r\sin kr}{k} dr \\
&= \frac{\sqrt{2\pi}(R_b^\pm)^3}{6} \frac{e^{-kR_b^\pm} - Be^{-B^* kR_b^\pm} - B^* e^{-B^* kR_b^\pm}}{kR_b^\pm} \\
&= \frac{\sqrt{2\pi}(R_b^\pm)^3}{6} f[kR_b^\pm], \qquad (2)
\end{aligned}
$$

with $B = (1+i\sqrt{3})/2$ and

$$
\begin{aligned}
\tilde{\alpha}^\pm(k) &= \mathcal{F}\left[\alpha_1^\pm(r)\right] \\
&= \frac{g\Omega_c^2}{2i\Gamma(2i\Gamma\delta - \Omega_c^2)} \mathcal{F}\left[\frac{1}{1+(r/R_b^\pm)^6}\right] \pm \frac{g\Omega_c^2}{2i\Gamma(2i\Gamma\delta - \Omega_c^2)} \mathcal{F}\left[\frac{1}{1+(r/R_b^\pm)^6}\right] \qquad (3) \\
&= \frac{g\Omega_c^2}{2i\Gamma(2i\Gamma\delta - \Omega_c^2)} \frac{\sqrt{2\pi}}{6} \left((R_b^+)^3 f[kR_b^+] \pm (R_b^-)^3 f[kR_b^-]\right).
\end{aligned}
$$

Because

$$
\begin{aligned}
\alpha^+(0) &= \frac{g\Omega_c^2}{2i\Gamma(2i\Gamma\delta - \Omega_c^2)} \frac{\sqrt{2\pi}}{6} \lim_{k\to 0} \left((R_b^+)^3 f[kR_b^+] + (R_b^-)^3 f[kR_b^-]\right) \\
&= \frac{g\Omega^2\sqrt{2\pi}}{12i\Gamma(2i\Gamma\delta - \Omega^2)} \left((R_b^+)^3 + (R_b^-)^3\right), \qquad (4)
\end{aligned}
$$

so

$$
\begin{aligned}
\tilde{\chi}_l &= A_0 + \sqrt{2\pi} n \tilde{\alpha}^+(0) \\
&= A_0 + A_1 \left((R_b^+)^3 f[kR_b^+] \pm (R_b^-)^3 f[kR_b^-]\right), \qquad (5)
\end{aligned}
$$

with $A_1 = \pi g \Omega_c^2 n / (6i\Gamma(2i\Gamma\delta - \Omega_c^2))$. Obviously, $\tilde{\chi}_l$ is k independent.

For the the nonlocal term:

$$
\begin{aligned}
& \mathcal{F}\left[\int \chi_n(\mathbf{r}-\mathbf{r}') E(\mathbf{r}') d\mathbf{r}'\right] \\
&= \mathcal{F}\left[\int_{-\infty}^{\infty} g\alpha_n(\mathbf{r}-\mathbf{r}') e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')} n(\mathbf{r}') E(\mathbf{r}') d\mathbf{r}'\right] \\
&= \mathcal{F}\left[\alpha^-(\mathbf{r}) e^{i\mathbf{K}\cdot\mathbf{r}} * n(\mathbf{r}) E(\mathbf{r})\right] \\
&= \sqrt{2\pi} \mathcal{F}\left[\alpha^-(\mathbf{r}) e^{i\mathbf{K}\cdot\mathbf{r}}\right] \mathcal{F}\left[n(\mathbf{r}) E(\mathbf{r})\right] \\
&= \sqrt{2\pi} n \tilde{\alpha}^-(\mathbf{k}+\mathbf{K}) \tilde{E}(\mathbf{k}) \\
&= \tilde{\chi}_n(\mathbf{k}) \tilde{E}(\mathbf{k}) \qquad (6)
\end{aligned}
$$

with $\tilde{\chi}_n(\mathbf{k}) = A_1\left[(R_b^+)^3 f(|\mathbf{k}+\mathbf{K}|R_b^+) - (R_b^-)^3 f(|\mathbf{k}+\mathbf{K}|R_b^-)\right]$, which is k dependent.