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Photon–phonon spin–orbit interaction in optical fibers: supplement

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The main goal of this chapter is to present the method of obtaining the perturbed optical modes in the vicinity of the acousto - optic resonance using the perturbation approach to the waveguide equation.

1. THE OPTICAL WAVEGUIDE EQUATION IN THE PRESENCE OF A TRANSVERSE CIRCULARLY POLARIZED FLEXURAL ACOUSTIC WAVE

Let us consider the fiber with permittivity implied in Eq. (2). As is well known, the propagation of the light beam through a dielectric medium can be described by the vector wave equation:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\epsilon(\mathbf{r},t)\mathbf{E}\right] = -\nabla\big(\mathbf{E}\cdot\nabla\ln\epsilon(\mathbf{r},t)\big)\,. \tag{S1}$$

where $\nabla = (\partial x, \partial y, \partial y)$, c is the speed of light in vacuum, $\mathbf{E}$ is the electric field. Since one has the following estimation for derivatives with respect to time

$$\epsilon\frac{\partial \mathbf{E}}{\partial t} \propto \epsilon_{co}\omega\mathbf{E}, \quad \mathbf{E}\frac{\partial \epsilon}{\partial t} \propto \epsilon_{co}\Delta\frac{u_0}{r_0}\Omega t\,, \tag{S2}$$

where ω and Ω is the frequency of optical and acoustical waves, respectively, and allowing for experimentally justified relations $\Omega/\omega \ll 1$ and $r_0/u_0 \gg 1$, one can disregard the derivative of $\epsilon(\mathbf{r},t)$ with respect to time in Eq. (S1):

$$\frac{\partial^2}{\partial^2 t}\left[\epsilon(\mathbf{r},t)\mathbf{E}\right] \approx \epsilon(\mathbf{r},t)\frac{\partial^2 \mathbf{E}}{\partial t^2}\,. \tag{S3}$$

The gradient term on the right - hand side of Eq. (S1), which describes the spin-orbit interaction (SOI) of light [1], for weakly guiding fibers with $\Delta \sim 10^{-2} - 10^{-3}$, $kr_0 \gg 1$ and $E_t \gg E_z$ has the order of $(\Delta/r_0^2)E_t$, where Δ is the optical contrast between core and cladding, r_0 is the fiber core's radius, E_t is the transverse component of the electric field and $k = \omega/c$. Taking into account that the order of the acoustically - induced term in Eq. (S1) is $k^2\Delta u_0 \varepsilon_{\mathrm{co}} E_t/r_0$, where u_0 is the acoustic amplitude, one can disregard the effect of the SOI of light in a practically realizable case $\varepsilon_{\mathrm{co}} k^2 u_0 r_0 \gg 1$. Using Eq. (S3) enables one to obtain the following scalar wave equation in the transverse field:

$$\left[\nabla^2 - \frac{\epsilon(\mathbf{r},t)}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{E}_{\mathrm{t}}(\mathbf{r},t) = 0\,. \tag{S4}$$

Using the standard ansatz:

$$\mathbf{E}_{\mathrm{t}}(\mathbf{r},t) = \sum_{-\infty}^{\infty}\tilde{\mathbf{e}}_m(r,\varphi)\cdot e^{i\left[(\beta+mK)z-(\omega+m\Omega)t\right]}\,, \tag{S5}$$

where β is the desired propagation constant and K is the acoustic wave - vector, one can exclude t and z variables in the scalar wave equation, thus obtaining the following infinite set of equations in amplitudes $\tilde{\mathbf{e}}_m$:

$$\left[\nabla_t^2 + \epsilon_0 k_m^2 - (\beta+mK)^2\right]\tilde{\mathbf{e}}_m + \frac{1}{2}\left(k_{m-1}^2 e^{i\Sigma\varphi}\tilde{\mathbf{e}}_{m-1} + k_{m+1}^2 e^{-i\Sigma\varphi}\tilde{\mathbf{e}}_{m+1}\right) = 0\,, \tag{S6}$$

where $\nabla_t = (\partial/\partial x, \partial/\partial y)$, $k_m = (\omega + m\Omega)/c$ and Σ specifies the handedness of acoustic circular polarization (acoustic spin angular momentum (SAM)). Introducing the state - vector $|\Psi\rangle = \sum_{-\infty}^{\infty}\tilde{\mathbf{e}}_m(r,\varphi)|m\rangle$, with $|m\rangle = (...,0,1,0,...)^{\mathrm{T}}$, where the unity is placed at the m - th position, T stands for transposition, Eq. (S6) can be brought to the form of an eigenvalue equation:

$$(\hat{H}_0 + \hat{V}_{\mathrm{AOI}})|\Psi\rangle = \beta^2|\Psi\rangle\,, \tag{S7}$$

where

$$\hat{H}_0 = \sum_{-\infty}^{\infty} \left(\nabla_t^2 + \epsilon_0 k_m^2 - 2mK\beta - m^2K^2\right)|m\rangle\langle m| \,, \tag{S8}$$

and

$$\hat{V}_{\text{AOI}} = \xi \sum_{n=-\infty}^{\infty} \left(k_{n-1}^2 e^{i\Sigma\varphi}|n\rangle\langle n-1| + k_{n+1}^2 e^{-i\Sigma\varphi}|n\rangle\langle n+1|\right), \tag{S9}$$

where $\xi = \epsilon_{co}\Delta\frac{u_0}{r_0}f'$, $f(r)$ is the fiber's profile function [2], the prime stands for the derivative with respect to the argument .The first operator $\hat{H}_0$ describes the optical modes of the unperturbed circular fiber, while the second one $\hat{V}_{\text{AOI}}$ accounts for the acousto - optic interaction. Since $\hat{H}_0 \propto k^2\varepsilon$ and $\hat{V}_{AOI} \propto k^2\Delta u_0\epsilon_{\text{co}}/r_0$, it follows $\hat{V}_{\text{AOI}}/\hat{H}_0 \propto \Delta u_0/r_0 \ll 1$, so that for all practically reasonable parameters of the system one can use the perturbation theory for solving Eq. (S7).

2. RESONANCE PERTURBATION THEORY AND RESONANCE OPTICAL FIBER MODES

The zero -order eigenvalue equation, $(\hat{H}_0 - \overline{\beta}^2)|\overline{\Psi}\rangle = 0$, or, equivalently,

$$\left(\nabla_t^2 + \epsilon_0 k_m^2\right)\tilde{\mathbf{e}}_m = (\overline{\beta} + mK)^2\tilde{\mathbf{e}}_m \,, \tag{S10}$$

yields the standard optical modes of the unperturbed circular fibers with all the possible frequencies and azimuthal numbers, which can be chosen in the form of circularly polarized optical vortices (OVs) $|m,\sigma,\ell\rangle$:

$$|m,\sigma,\ell\rangle = F_\ell(r)\exp(i\ell\varphi)(1,i\sigma)^{\text{T}}|m\rangle \,, \tag{S11}$$

Here m determines the optical frequency of the OV through $\omega_{\text{m}} = \omega + m\Omega$, $\sigma = \pm 1$ indicates the sign of optical circular polarization, ℓ is an integer topological charge and $F_\ell(r)$ is the known radial function [2], which is expressed through the Bessel functions of the first kind in the core and the modified Bessel functions in the cladding for the implied here step - index fibers. The radial number is omitted. Importantly, the corresponding propagation constants:

$$\overline{\beta}_{m,\ell}(K) = \tilde{\beta}_\ell - mK. \tag{S12}$$

where $\tilde{\beta}_\ell$ is the known scalar propagation constant [2], prove to be modified so that they become dependent on the acoustic K - vector. The spectrum in Eq. (S12) is degenerate at resonance values of K, where the corresponding spectral curves intersect (see Fig. (S1)):

$$\overline{\beta}_{m,\ell}(\overline{K}) = \overline{\beta}_{m',\ell'}(\overline{K}) \,. \tag{S13}$$

Here the resonance values of acoustic wave number are given by:

$$\overline{K} = \frac{\tilde{\beta}_\ell - \tilde{\beta}_{\ell'}}{m - m'} \,. \tag{S14}$$

This kinematic condition determines the unperturbed optical states $|m,\sigma,\ell\rangle$ and $|m',\sigma',\ell'\rangle$ that could be coupled efficiently by the acoustic perturbation (S9). Further we assume a few-mode fiber that supports the propagation of only the modes with $|\ell| = 0,1$. For such a system there exist three resonance points (see Fig. (S1)). In the vicinity of such resonance points, a highly efficient coupling of the corresponding zero-approximation modes takes place provided the dynamic condition is fulfilled:

$$\langle m,\sigma,\ell|\hat{V}_{\text{AOI}}|m',\sigma',\ell'\rangle = \eta\delta_{\sigma',\sigma}\left(k_{m-1}^2\delta_{m,m'-1}\delta_{\ell-\ell',-\Sigma} + k_{m+1}^2\delta_{m,m'+1}\delta_{\ell-\ell',\Sigma}\right) \neq 0 \,, \tag{S15}$$

Here the standard scalar product with the integration over the total transverse cross - section of the fibre is implied, $\eta = \frac{\Delta u_0\epsilon_{co}}{r_0\sqrt{\int_0^\infty RF_\ell^2 dR}\sqrt{\int_0^\infty RF_{\ell'}^2 dR}}$ and $\delta_{l,n}$ is the Kronecker symbol. This condition entails the following selection rules that guarantee the coupling of unperturbed fiber modes:

$$\sigma' = \sigma,\ m' = m \pm 1,\ \ell' = \ell \pm \Sigma \,. \tag{S16}$$

Here the first expression is due to the scalar nature of the AOI - operator in Eq. (S9), the second one corresponds to the shift of the incident optical frequency, while the last one describes hybridization of the acoustic spin and optical orbital degrees of freedom. From Eq. (S16) one can readily see that the optical states near the second resonance point "C" in Fig. (S1) do not obey the frequency selection rule and thus are not coupled by the perturbation. On the contrary, identically polarized states with $\sigma = \pm 1$ in the vicinity of point "A"

$$|0,\sigma,0\rangle, |-1,\sigma,-\Sigma\rangle\,, \tag{S17}$$

and states in the vicinity of point "B"

$$|0,\sigma,-\Sigma\rangle, |+1,\sigma,0\rangle, . \tag{S18}$$

obey all the above given selection rules, thereby forming the two sets of basis for the perturbation theory. Please note that these two resonances occur at the same value of the acoustic $\overline{K}$ - vector :

$$\overline{K} = \tilde{\beta}_0 - \tilde{\beta}_1\,. \tag{S19}$$

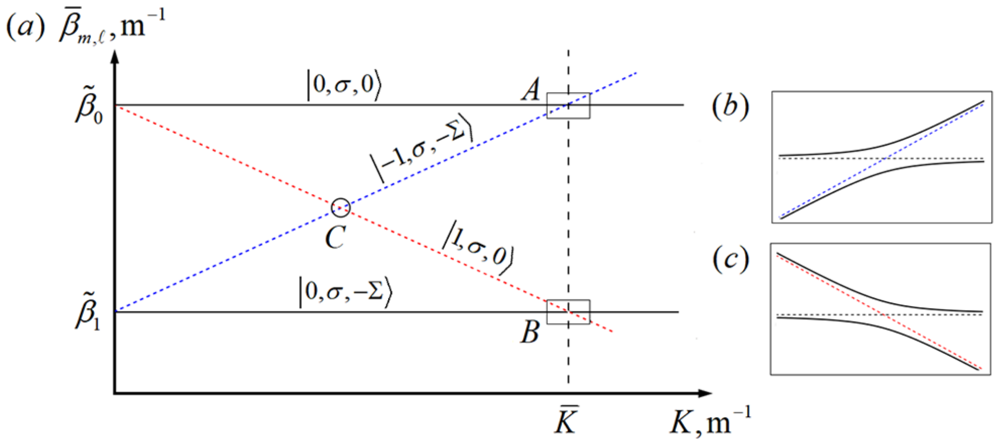

Fig. S1. (Color online) Inset (a) shows zero- -order propagation constants in Eq. (S12) as a function of the acoustic K- -vector.The points of their intersection are the resonance points, in the vicinity of which even a small perturbation can cause a strong coupling of the corresponding zero-order modes $|m,\sigma,\ell\rangle$. The states near resonance points A and B, unlike the modes in the vicinity of point C, obey both the kinematic and dynamic resonance conditions. Insets (b) and (c) demonstrate the anticrossing of spectral branches due to hybridization of zero-order modes near resonance points A and B, respectively.

As is known [3, 4], to establish the structure of perturbed fiber modes near the points of accidental degeneracy, one has to build the matrix H_{ij} of the total operator in wave equation (S7) over the basis of such eigenvectors of $\hat{H}_0$ that obey both the kinematic and dynamical conditions.

In this way, using Eq. (S15) and allowing for:

$$\langle m,\sigma,\ell|\hat{H}_0|m',\sigma',\ell'\rangle = [\tilde{\beta}_\ell^2 - (\beta + mK)^2]\delta_{m',m}\delta_{\sigma',\sigma}\delta_{\ell',\ell}\,, \tag{S20}$$

one can show that near the first resonance "A" the matrices built over two- -dimensional subspaces of the right and left circular polarizations have the same form:

$$\hat{H}^A = \begin{pmatrix} \tilde{\beta}_0^2 - \beta^2 & \alpha_0 \\ \alpha_{-1} & \tilde{\beta}_1^2 - (\beta - K)^2 \end{pmatrix}, \tag{S21}$$

where $\alpha_k = k_m^2\eta$. The solutions $\mathbf{x} = (x1, x2)^{\mathrm{T}}$ of the eigenvalue equation:

$$\hat{H}\mathbf{x} = 0\,, \tag{S22}$$

give the desired resonance optical modes through $|\Psi^\sigma\rangle = x1|0,\sigma,0\rangle + x2|-1,\sigma,-\Sigma\rangle$. Linearizing the eigenvalue equation near the resonance point $[\overline{K},\tilde{\beta}_0]$:

$$\begin{pmatrix} -2\tilde{\beta}_0\delta_0 & \alpha_0 \\ \alpha_0 & 2\tilde{\beta}_1(\varepsilon - \delta_0) \end{pmatrix}\mathbf{x} = 0\,, \tag{S23}$$

where the obvious approximation $\alpha_0 \approx \alpha_{-1}$ is used and the following detunings are introduced:

$$\varepsilon = K - \overline{K},\ \delta_\ell = \beta - \tilde{\beta}_\ell\,, \tag{S24}$$

we arrive at the following expressions for the resonance acoustically - driven fiber modes:

$$\begin{aligned} |\Psi_1^{(\sigma)}\rangle &= \Big[\sin\theta|0,\sigma,0\rangle + \cos\theta|-1,\sigma,-\Sigma\rangle\Big]e^{i\beta_1 z}\,, \\ |\Psi_2^{(\sigma)}\rangle &= \Big[\cos\theta|0,\sigma,0\rangle - \sin\theta|-1,\sigma,-\Sigma\rangle\Big]e^{i\beta_2 z}\,. \end{aligned} \tag{S25}$$

As is seen, the modes in Eq. (S25) are linear combinations of the identically circularly polarized Gauss-like fundamental mode and frequency down - shifted OV with topological charge $-\Sigma$. The energy distribution within the hybrid modes is governed by the parameter $0 < \theta \leqslant \pi/4$ defined as $\cos 2\theta = \epsilon/\sqrt{\epsilon^2 + Q^2}$. The parameter Q, which determine the coupling strength between zero-order modes, equals to

$$Q = \frac{k_0^2 \Delta\epsilon_{co} u_0}{r_0 \tilde{\beta}_0 \sqrt{\int_0^\infty RF_\ell^2 dR \int_0^\infty RF_{\ell'}^2 dR}}\,. \tag{S26}$$

The propagation constants of modes (S25) are found to be:

$$\beta_{1,2} = \tilde{\beta}_0 + \epsilon_\pm\,, \tag{S27}$$

where $\epsilon_\pm = (1/2)(\epsilon \pm \sqrt{\epsilon^2 + Q^2})$.

Analogously, for the resonance point "B" the matrix looks like:

$$\hat{H}^B = \begin{pmatrix} \tilde{\beta}_1^2 - \beta^2 & \alpha_1 \\ \alpha_0 & \tilde{\beta}_0^2 - (\beta + K)^2 \end{pmatrix}, \tag{S28}$$

and the corresponding eigenvalue equation linearized near the point $[\overline{K}, \tilde{\beta}_1]$ reads as

$$\begin{pmatrix} -2\tilde{\beta}_1 \delta_1 & \alpha_0 \\ \alpha_0 & -2\tilde{\beta}_1(\varepsilon + \delta_1) \end{pmatrix} \mathbf{x} = 0\,, \tag{S29}$$

which gibes rise to the second set of resonance fiber modes:

$$\begin{aligned} |\Psi_3^{(\sigma)}\rangle &= \Big[\sin\theta|0,\sigma,-\Sigma\rangle - \cos\theta|+1,\sigma,0\rangle\Big]e^{i\beta_3 z}\,, \\ |\Psi_4^{(\sigma)}\rangle &= \Big[\cos\theta|0,\sigma,-\Sigma\rangle + \sin\theta|+1,\sigma,0\rangle\Big]e^{i\beta_4 z}\,. \end{aligned} \tag{S30}$$

The corresponding propagation constants are:

$$\beta_{3,4} = \tilde{\beta}_1 - \epsilon_\pm\,, \tag{S31}$$

It is easily seen that the modes in Eq. (S30) are superpositions of the identically polarized OV of topological charge $-\Sigma$ and frequency up - shifted Gauss-like fundamental mode. Naturally, at the resonance $\epsilon = 0$ both the strongest splitting of the propagation constants $\pm Q$ in Eqs. (S27) and (S31) (see the inset (a) and (b) in Fig. (S1)) and the strongest hybridization of the partial states within modes in Eqs. (S25) and (S30) at $\theta = \pi/4$ take place.

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