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# Focusing light inside dynamic scattering media with millisecond digital optical phase conjugation: supplementary material

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**This document provides supplementary information to "Focusing light inside dynamic scattering media with millisecond digital optical phase conjugation," https://doi.org/10.1364/optica.4.000280. It shows the derivation of the theoretical peak-to-background ratio (PBR) of binary-phase modulation based wavefront shaping. It also shows the derivation of the relationship between the PBR and the speckle correlation time when there are multiple input modes. In addition, the binary phasemaps displayed on the ferroelectric spatial light modulator for focusing light through or inside scattering media are provided, and the code for calculating the pattern is shown at the end.** 

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## 1. SUPPLEMENTARY NOTES

### A. Peak-to-background ratio of binary-phase modulation based wavefront shaping

Here, we derive the theoretical peak-to-background ratio (PBR) of the focus when the playback wavefront is achieved by a ferroelectric liquid crystal based spatial light modulator (FLC-SLM) that performs binary-phase modulation. For a thick scattering medium, the incident light field $\mathbf{E_{in}}$ and the scattered light field $\mathbf{E_S}$ are connected by a transmission matrix $\mathbf{T}=\left(a_{ij}\right)_{N\times S}$, whose element $a_{ij}$ follows a circular Gaussian distribution. For an incident light field $\mathbf{E_{in}}=(1\ 0\ \cdots\ 0)^T_{1\times S}$ that has only one nonzero element (one input mode [1]), the scattered light field is computed as $\mathbf{E_S}=\mathbf{TE_{in}}=\left(a_{11}\ a_{21}\ \cdots\ a_{N1}\right)^T$, where the upper case $T$ stands for matrix transposition. After the scattered light field is measured, the playback field under binary-phase modulation takes the following form: $\mathbf{E_P}=\mathrm{BP}(\mathbf{E_S^*})=\left(\mathrm{BP}\left(a_{11}^*\right)\ \mathrm{BP}\left(a_{21}^*\right)\ \cdots\ \mathrm{BP}\left(a_{N1}^*\right)\right)^T$, where $*$ denotes complex conjugation, and the binary-phase operator $\mathrm{BP}\left(z=x+iy\right)$ is defined as

$$\mathrm{BP}\left(z\right)=\begin{cases}e^{i0}=1, & \text{if } y\geq 0\\ e^{i\pi}=-1, & \text{if } y<0\end{cases}. \tag{S1}$$

After passing through the scattering medium in the backward direction, the phase conjugated field becomes $\mathbf{E_{OPC}}=\mathbf{T}^T\mathbf{E_P}$. Among all the elements of $\mathbf{E_{OPC}}$, the first element represents the electric field at the focus, which can be calculated by

$$E_{\text{peak}}=\sum_{m=1}^{N}a_{m1}\mathrm{BP}\left(a_{m1}\right). \tag{S2}$$

Thus, the intensity of the focus is calculated by

$$\begin{aligned}
I_{\text{peak}}&=\left\langle\left|E_{\text{peak}}\right|^2\right\rangle\\
&=\left|\frac{N}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\left(x+iy\right)\left[\theta\left(y\right)-\theta\left(-y\right)\right]e^{-\left(x^2+y^2\right)/\left(2\sigma^2\right)}\right|^2,\\
&=\left|\frac{iN}{2\pi\sigma^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dxdy\left[y\theta\left(y\right)-y\theta\left(-y\right)\right]e^{-\left(x^2+y^2\right)/\left(2\sigma^2\right)}\right|^2,\\
&=\left|\frac{2iN}{\sqrt{2\pi\sigma^2}}\int_{0}^{+\infty}dyye^{-y^2/\left(2\sigma^2\right)}\right|^2,\\
&=\left|\frac{2iN\sigma}{\sqrt{2\pi}}\Gamma\left(1\right)\right|^2=\frac{2}{\pi}\sigma^2N^2,
\end{aligned} \tag{S3}$$

where $\theta\left(\cdot\right)$ is the Heaviside step function, $\Gamma\left(x\right)=\int_0^{+\infty}t^{x-1}e^{-t}dt$ is the gamma function, and $\left\langle\bullet\right\rangle$ denotes ensemble averaging. In the above derivation, we have converted the summation into an integration. Moreover, for the matrix element $a_{ij}$, we have used the fact that both its real part ($x$) and its imaginary part ($y$) obey the same Gaussian distribution with a mean of zero and a standard deviation of $\sigma$.

The remaining elements of $\mathbf{E_{OPC}}$ represent the fields of the background, and are calculated by $E_{\text{background}} = \sum_{m=1}^{N} a_{mn} \mathrm{BP}(a_{m1}),\ n \neq 1$. Thus, the background intensity can be obtained by

$$\begin{aligned}
I_{\text{background}} &= \left\langle \left| \sum_{m=1}^{N} a_{mn} \mathrm{BP}(a_{m1}) \right|^2 \right\rangle, n \neq 1, \\
&= N \mathrm{Var}\left(a_{mn} \mathrm{BP}(a_{m1})\right), \\
&= N \mathrm{Var}(a_{mn}) \mathrm{Var}\left(\mathrm{BP}(a_{m1})\right), \\
&= N \times 2\sigma^2 \times 1 \\
&= 2\sigma^2 N,
\end{aligned} \tag{S4}$$

where $\mathrm{Var}(\bullet)$ sums the variances of both the real part and the imaginary part of a random variable. Combining the results given by Eqs. (S3) and (S4), we obtain the theoretical PBR of the focus achieved with binary-phase modulation:

$$\mathrm{PBR}_1 = I_{\text{peak}} / I_{\text{background}} = \frac{1}{\pi} N. \tag{S5}$$

When there are $M$ input modes, it can be proved that the theoretical PBR is scaled down by a factor of $M$:

$$\mathrm{PBR}_{\mathrm{M}} = \frac{1}{\pi} \frac{N}{M}. \tag{S6}$$

When a scattering medium is sufficiently thick, polarization can be completely scrambled by scattering. However, DOPC systems, except the one presented in Ref. [2], phase-conjugates only a single polarization of the sample light. In this case, by using a vector transmission matrix in the derivation [2, 3], we can find that the PBRs derived in Eqs. (S5) and (S6) are reduced by half. Such a PBR reduction can be understood by considering the background elevation due to the field polarized orthogonally to the incident polarization direction.

**B. Relationship between the PBR and the speckle correlation time when there are multiple input modes**

For dynamic scattering media, the influence of speckle decorrelation on the PBR of a DOPC-achieved focus was studied in Ref. [4]. However, only a single input mode was considered in the authors' derivation. Since time-reversed ultrasonically encoded (TRUE) optical focusing typically encounters a large number of input modes $M$, we extend the previous work to the case of $M$ greater than 1. We also derive the relationship between the PBR and the speckle correlation time $\tau_c$.

Consider an incident field $\mathbf{E_{in}}(t) = (p_1\ p_2\ \cdots\ p_M\ 0\ \cdots\ 0)^T_{S\times 1}$ which has $M$ non-zero elements. After passing through a thick scattering medium, the scattered light field $\mathbf{E_S}(t)$ can be calculated by

$$\mathbf{E_S}(t) = \mathbf{T}(t)\mathbf{E_{in}}(t) = \begin{pmatrix} a_{11}(t)\ a_{12}(t)\ \cdots a_{1S}(t) \\ a_{21}(t)\ a_{22}(t)\ \cdots a_{2S}(t) \\ \vdots \quad \vdots \quad \vdots \\ a_{N1}(t)\ a_{N2}(t) \cdots\ a_{NS}(t) \end{pmatrix}_{N\times S} \begin{pmatrix} p_1 \\ \vdots \\ p_M \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{S\times 1}$$

$$= \begin{pmatrix} a_{11}(t)p_1 + a_{12}(t)p_2 + \cdots + a_{1M}(t)p_M \\ a_{21}(t)p_1 + a_{22}(t)p_2 + \cdots + a_{2M}(t)p_M \\ \vdots \\ a_{N1}(t)p_1 + a_{N2}(t)p_2 + \cdots + a_{NM}(t)p_M \end{pmatrix}_{N\times 1}. \tag{S7}$$

Hence, the first-order correlation function of the light field of the $i$-th output mode can be calculated by

$$g^{(1)}(\tau) = \frac{\left\langle E^*(t)E(t+\tau)\right\rangle_t}{\sqrt{\left\langle |E(t)|^2\right\rangle_t}\sqrt{\left\langle |E(t+\tau)|^2\right\rangle_t}}$$

$$= \frac{\left\langle \left[a_{i1}(t)p_1 + a_{i2}(t)p_2 + \cdots + a_{iM}(t)p_M\right]^* \left[a_{i1}(t+\tau)p_1 + a_{i2}(t+\tau)p_2 + \cdots + a_{iM}(t+\tau)p_M\right]\right\rangle_t}{\left\langle \left[a_{i1}(t)p_1 + a_{i2}(t)p_2 + \cdots + a_{iM}(t)p_M\right]^* \left[a_{i1}(t)p_1 + a_{i2}(t)p_2 + \cdots + a_{iM}(t)p_M\right]\right\rangle_t}, \tag{S8}$$

where $\langle\bullet\rangle_t$ takes an ensemble average over time, and we assume the random process is stationary and ergodic. Since $\langle a_{ij}(t)a_{ik}(t)\rangle_t = \langle a_{ij}(t)a_{ik}(t+\tau)\rangle_t = 0$ for any $j \neq k$, we have

$$g^{(1)}(\tau) = \frac{\left\langle \left[a_{i1}^*(t)a_{i1}(t+\tau)|p_1|^2 + a_{i2}^*(t)a_{i2}(t+\tau)|p_2|^2 + \cdots + a_{iM}^*(t)a_{iM}(t+\tau)|p_M|^2\right]\right\rangle_t}{\left\langle \left[a_{i1}^*(t)a_{i1}(t)|p_1|^2 + a_{i2}^*(t)a_{i2}(t)|p_2|^2 + \cdots + a_{iM}^*(t)a_{iM}(t)|p_M|^2\right]\right\rangle_t}$$

$$\begin{aligned}
&= \frac{\frac{1}{M}\left(|p_1|^2 + |p_2|^2 + \cdots + |p_M|^2\right) M\, \overline{a^*(t)a(t+\tau)}}{\frac{1}{M}\left(|p_1|^2 + |p_2|^2 + \cdots + |p_M|^2\right) M\, \overline{a^*(t)a(t)}} \\
&= \frac{\overline{a^*(t)a(t+\tau)}}{\overline{a^*(t)a(t)}},
\end{aligned} \tag{S9}$$

where $\overline{A}$ denotes the mean of random variable $A$, and $a(t)$ denotes a random variable that follows a circular Gaussian distribution. For the second equality, we used the fact that $|p|^2$ and $a^*(t)a(t+\tau)$ are independent.

After passing through the scattering medium, the phase-conjugated light field at $t+\tau$ is computed as

$$\mathbf{E_{OPC}}(t+\tau) = \left[\mathbf{T}(t+\tau)\right]^T \mathbf{E_S}^*(t)$$

$$= \begin{pmatrix} a_{11}(t+\tau) & a_{21}(t+\tau) & \cdots & a_{N1}(t+\tau) \\ a_{12}(t+\tau) & a_{22}(t+\tau) & \cdots & a_{N2}(t+\tau) \\ \vdots & \vdots & & \vdots \\ a_{1S}(t+\tau) & a_{2S}(t+\tau) & \cdots & a_{NS}(t+\tau) \end{pmatrix} \begin{pmatrix} a_{11}(t)p_1 + a_{12}(t)p_2 + \cdots + a_{1M}(t)p_M \\ a_{21}(t)p_1 + a_{22}(t)p_2 + \cdots + a_{2M}(t)p_M \\ \vdots \\ a_{N1}(t)p_1 + a_{N2}(t)p_2 + \cdots + a_{NM}(t)p_M \end{pmatrix}^*.$$

(S10)

So, the $i$-th element of $\mathbf{E_{OPC}}(t+\tau)$ can be calculated by

$$E_{\mathrm{OPC}i}(t+\tau) = (a_{1i}(t+\tau)\ a_{2i}(t+\tau)\ \cdots\ a_{Ni}(t+\tau)) \begin{pmatrix} \sum_{j=1}^{M} a_{1j}^*(t)p_j^* \\ \sum_{j=1}^{M} a_{2j}^*(t)p_j^* \\ \vdots \\ \sum_{j=1}^{M} a_{Nj}^*(t)p_j^* \end{pmatrix}$$

$$= \sum_{k=1}^{N} a_{ki}(t+\tau) \sum_{j=1}^{M} a_{kj}^*(t)p_j^* = \sum_{k=1}^{N}\sum_{j=1}^{M} a_{ki}(t+\tau)a_{kj}^*(t)p_j^*. \tag{S11}$$

Then, we have

$$\left\langle E_{\mathrm{OPC}i}(t+\tau)\right\rangle_t = \left\langle \sum_{k=1}^{N} a_{ki}(t+\tau)a_{ki}^*(t)p_i^*\right\rangle_t = p_i^* N\, \overline{a^*(t)a(t+\tau)}. \tag{S12}$$

Thus, the ratio of the PBRs at time $t$ and $t+\tau$ can be calculated by

$$\begin{aligned}
\frac{\mathrm{PBR}(t+\tau)}{\mathrm{PBR}(t)} = \frac{\sum_{i=1}^{M} I_{\mathrm{OPC}i}(t+\tau)}{\sum_{i=1}^{M} I_{\mathrm{OPC}i}(t)} &= \frac{\sum_{i=1}^{M}\left|\left\langle E_{\mathrm{OPC}i}(t+\tau)\right\rangle_t\right|^2}{\sum_{i=1}^{M}\left|\left\langle E_{\mathrm{OPC}i}(t)\right\rangle_t\right|^2} \\
&= \left|\frac{\overline{a^*(t)a(t+\tau)}}{\overline{a^*(t)a(t)}}\right|^2.
\end{aligned} \tag{S13}$$

Based on Eqs. (S9) and (S13), we get

$$\frac{\mathrm{PBR}(t+\tau)}{\mathrm{PBR}(t)}=\left|g^{(1)}(\tau)\right|^{2}. \quad \text{(S14)}$$

On the other hand, the correlation coefficient $R_{\mathrm{I}}$ of two speckle patterns $I(\vec{r},t)$ and $I(\vec{r},t+\tau)$, recorded at $t$ and $t+\tau$, is defined as

$$R_{\mathrm{I}}(\tau)=\frac{\left\langle\left(I(\vec{r},t)-\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right)\left(I(\vec{r},t+\tau)-\left\langle I(\vec{r},t+\tau)\right\rangle_{\vec{r}}\right)\right\rangle_{\vec{r}}}{\sqrt{\left\langle\left[I(\vec{r},t)-\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right]^{2}\right\rangle_{\vec{r}}}\sqrt{\left\langle\left[I(\vec{r},t+\tau)-\left\langle I(\vec{r},t+\tau)\right\rangle_{\vec{r}}\right]^{2}\right\rangle_{\vec{r}}}},$$

(S15)

where $\langle\bullet\rangle_{\vec{r}}$ takes an average over all the pixels. We assume the random process is stationary, then,

$$\begin{aligned}R_{\mathrm{I}}(\tau)&=\frac{\left\langle\left(I(\vec{r},t)-\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right)\left(I(\vec{r},t+\tau)-\left\langle I(\vec{r},t+\tau)\right\rangle_{\vec{r}}\right)\right\rangle_{\vec{r}}}{\left\langle\left[I(\vec{r},t)-\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right]^{2}\right\rangle_{\vec{r}}}\\&=\frac{\left\langle\left(I(\vec{r},t)-\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right)\left(I(\vec{r},t+\tau)-\left\langle I(\vec{r},t+\tau)\right\rangle_{\vec{r}}\right)\right\rangle_{\vec{r}}}{\left[\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right]^{2}}, \quad \text{(S16)}\end{aligned}$$

where the last equal sign holds because $\sqrt{\left\langle\left[I(\vec{r},t)-\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}\right]^{2}\right\rangle_{\vec{r}}}=\left\langle I(\vec{r},t)\right\rangle_{\vec{r}}$ for polarized fully developed speckles [5].

For a given output mode, the second-order correlation function of light is written as

$$\begin{aligned}g^{(2)}(\tau)&=\frac{\left\langle I(t)I(t+\tau)\right\rangle_{t}}{\left\langle I(t)\right\rangle_{t}\left\langle I(t+\tau)\right\rangle_{t}}=\frac{\left\langle I(t)I(t+\tau)\right\rangle_{t}}{\left[\left\langle I(t)\right\rangle_{t}\right]^{2}}\\&=\frac{\left\langle I(t)I(t+\tau)\right\rangle_{t}-\left[\left\langle I(t)\right\rangle_{t}\right]^{2}}{\left[\left\langle I(t)\right\rangle_{t}\right]^{2}}+1\\&=\frac{\left\langle\left(I(t)-\left\langle I(t)\right\rangle_{t}\right)\left(I(t+\tau)-\left\langle I(t+\tau)\right\rangle_{t}\right)\right\rangle_{t}}{\left[\left\langle I(t)\right\rangle_{t}\right]^{2}}+1. \quad \text{(S17)}\end{aligned}$$

Because of ergodicity, the ensemble average over time for the intensity of one output mode in Eq. (S17) can be calculated by averaging the intensities over all output modes (camera pixels). Then, from Eqs. (S16) and (S17), we have

$$g^{(2)}(\tau)=R_{\mathrm{I}}(\tau)+1. \quad \text{(S18)}$$

According to the Siegert relation, we have

$$g^{(2)}(\tau)=\beta\left|g^{(1)}(\tau)\right|^{2}+1, \quad \text{(S19)}$$

where $\beta$ is a constant depending on the experimental conditions. From Eqs. (S18) and (S19), we get

$$R_{\mathrm{I}}(\tau)=\beta\left|g^{(1)}(\tau)\right|^{2}. \quad \text{(S20)}$$

Based on the results in Eqs. (S14) and (S20), we obtain

$$\frac{\mathrm{PBR}(t+\tau)}{\mathrm{PBR}(t)}=\frac{1}{\beta}R_{\mathrm{I}}(\tau). \quad \text{(S21)}$$

Eq. (S21) shows that the PBR is proportional to the correlation coefficient $R_{\mathrm{I}}$ of speckle patterns. From Eq. (S18), for polarized fully developed speckles, $R_{\mathrm{I}}(t,\tau_{\mathrm{c}})=g^{(2)}(\tau)-1=\exp(-2t^{2}/\tau_{\mathrm{c}}^{2})$ [6-8], so we get

$$\mathrm{PBR}(t,\tau_{\mathrm{c}})=A\exp(-2t^{2}/\tau_{\mathrm{c}}^{2}), \quad \text{(S22)}$$

where $A=\mathrm{PBR}(t/\tau_{\mathrm{c}}\to 0)/\beta$, and $\mathrm{PBR}(t/\tau_{\mathrm{c}}\to 0)$ is the PBR achieved when the system runtime is infinitely small compared with the speckle correlation time of the sample.

## 2. SUPPLEMENTARY FIGURES

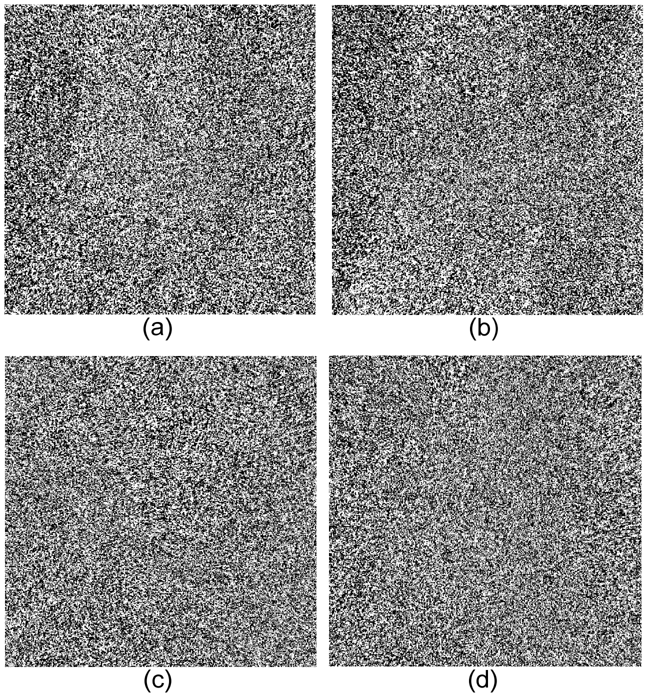


Fig. S1 Binary phasemaps displayed on the ferroelectric liquid crystal based spatial light modulator for focusing light through or inside scattering media. (a –b) The phasemaps used for focusing light through 3 mm thick chicken tissue when (a) the speckle correlation time ($\tau_{\mathrm{c}}$) is > 1 s, and (b) when $\tau_{\mathrm{c}}$ = 2 ms. (c – d) The phasemaps used for focusing light in between two pieces of 1 mm thick chicken tissue with a distance of 32 mm (c) when $\tau_{\mathrm{c}}$ > 1 s, and (d) when $\tau_{\mathrm{c}}$ = 5 ms. The ratios between the number of pixels whose phase values are 0 and the number of pixels whose phase values are π in (a – d) are 1.00, 0.98, 1.00, 1.00, respectively, which are expected for thick scattering media. The vague background feature in (a – b) is due to the compensation phasemap that corrects for the surface curvatures of the SLM and the reference beams.

## 3. SUPPLEMENTARY CODE

Here, we provide the code for the pattern calculation. However, we do not have permission from the companies to release the code for the control of the camera and the SLM.

```
CameraRecordTwoImages();

//Calculate the binary phasemap of the ultrasonically tagged light
//from the measured two holograms (wBuf1 and wBuf2).
for (unsigned long i = 0; i < numOfPixelsOfImage; i++)
{
    wBuf1[i]=(wBuf1[i]>wBuf2[i]);
}

//Calculate the pattern for the SLM: 1. Crop wBuf1 to 512 × 512
//using idx[], since the dimensions of the recorded image are 520 ×
//520. idx[] stores the index of each SLM pixel, and it is obtained
//before the experiment; 2. Compensate for the surface curvatures
//of the SLM and the reference beams, using the pre-calibrated
//binary image curvatureCompensation[]; 3. Scale the binary data
//to 0 and 255, since 0 and π are represented by 0 and 255 in the
//pattern transmitted to the 8-bit SLM controller.
```

```
for (unsigned long i=0; i<numOfPixelsOfSLM; i++)
{
    patternToSLM[i]=((wBuf1[idx[i]]+
curvatureCompensation[i])%2)*255;
}

TransferPatternToSLM(patternToSLM);
```

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